Exceptional times of the critical dynamical Erdős-Rényi graph

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Abstract

In this paper we introduce a network model which evolves in time, and study its largest connected component. We consider a process of graphs $(G_t : t \in [0,1])$, where initially we start with a critical Erdős-Rényi graph $\mathrm{ER}(n, 1/n)$, and then evolve forwards in time by resampling each edge independently at rate 1. We show that the size of the largest connected component that appears during the time interval $[0,1]$ is of order $n^{2/3}\log^{1/3} n$ with high probability. This is in contrast to the largest component in the static critical Erdős-Rényi graph, which is of order $n^{2/3}$.

1 Introduction and main result

An Erdős-Rényi graph $\mathrm{ER}(n,p)$ is a random graph on n vertices $\{1,\dots,n\}$, where each pair of vertices is connected by an edge with probability p, independently of all other pairs of vertices. Erdős and Rényi [7] introduced this graph (or rather a very closely related graph) and examined the structure of its connected components. Since then, Erdős-Rényi graphs have been intensively studied and have become a cornerstone of probability and combinatorics: see for example [4, 6, 15] and references therein.

Let L_n denote the largest connected component of an Erdős-Rényi graph $\mathrm{ER}(n,p)$ with $p = \mu/n$. We write $|L_n|$ for the number of vertices in L_n. This quantity exhibits a phase transition as μ passes 1:

(i) if $\mu < 1$, then $(\log n)^{-1}|L_n|$ converges in probability to $1/\alpha(\mu)$ where $\alpha(\mu) = \mu - 1 - \log\mu \in (0,\infty)$ (see [4, Corollaries 5.8 and 5.11]);

(ii) if $\mu = 1$, then $n^{-2/3}|L_n|$ converges in distribution to some non-trivial random variable as $n \to \infty$ (see [1]);

(iii) if $\mu > 1$, then $n^{-1}|L_n|$ converges in probability to $\theta(\mu)$ where $\theta(\mu) \in (0,1)$ is the unique solution to $\theta(\mu) = 1 - e^{-\mu\theta(\mu)}$ (see [15, Theorem 5.4]).

The model $\mathrm{ER}(n, 1/n)$ is therefore referred to as the critical Erdős-Rényi graph.

In this paper we study a dynamical version of the critical Erdős-Rényi graph, a process of random graphs $(G_t : t \in [0,1])$ on the vertex set $\{1, \ldots, n\}$, constructed as follows. Initially G_0 is distributed as $\mathrm{ER}(n, 1/n)$. Then the presence of each edge vw between vertices $v \neq w$ is resampled at rate 1, independently of all other edges. That is, at the times of a rate 1 Poisson process, we remove the edge vw if it exists, and then place an edge with probability $1/n$, independently of everything else. Clearly $\mathrm{ER}(n, 1/n)$ is invariant for this process, so for each $t \geq 0$, G_t is a realisation of $\mathrm{ER}(n, 1/n)$. Let $L_n(t)$ denote the largest connected component of G_t. Then for each fixed $t \in [0,1]$, $|L_n(t)|$ is of order $n^{2/3}$ with high probability as $n \to \infty$. Our main result gives a contrasting statement about the size of $\sup_{t\in[0,1]} |L_n(t)|$, showing that with high probability there are (rare) times when $|L_n(t)|$ is of order $n^{2/3} \log^{1/3} n$ (where we write $\log^\alpha n$ to mean $(\log n)^\alpha$).

Theorem 1.1. *As $n \to \infty$,*

$$\mathbb{P}\left(\frac{\sup_{t\in[0,1]} |L_n(t)|}{n^{2/3} \log^{1/3} n} > \beta\right) \longrightarrow \begin{cases} 1 & \text{if } \beta < 2/3^{2/3} \\ 0 & \text{if } \beta \geq 2/3^{1/3}. \end{cases}$$

1.1 Further discussion around Theorem 1.1

It is not difficult to deduce from known results (see for example [4]) together with a first moment method (see Section 5) that for Erdős-Rényi graphs away from criticality, the size of the biggest component in the dynamical model is of the same order as in the static model. That is, for $\mathrm{ER}(n, \mu/n)$ with $\mu < 1$, $\sup_{t\in[0,1]} |L_n(t)|$ is of order $\log n$ with high probability; and for $\mathrm{ER}(n, \mu/n)$ with $\mu > 1$, $\sup_{t\in[0,1]} |L_n(t)|$ is of order n with high probability. The critical graph $\mathrm{ER}(n, 1/n)$ is therefore the most interesting case.

Returning to $\mathrm{ER}(n, 1/n)$, the obvious open questions posed by Theorem 1.1 are:

- Does $\sup_{t\in[0,1]} |L_n(t)|/(n^{2/3} \log^{1/3} n)$ converge in probability as $n \to \infty$? If so, what is its limit?
- What does the set of exceptional times, i.e. $\{t \in [0,1] : |L_n(t)| \geq \beta n^{2/3} \log^{1/3} n\}$, look like?
- How does $\inf_{t\in[0,1]} |L_n(t)|$ behave?
- What if we resample each edge at rate n^γ for $\gamma \neq 0$?

For the first question we suspect that $\sup_{t\in[0,1]} |L_n(t)|/(n^{2/3} \log^{1/3} n) \to 2/3^{1/3}$ in probability as $n \to \infty$. We hope to address this in future work, but substantial further technical estimates are required.

We can say a limited amount about the second question. On the one hand, it is easy to check that the Lebesgue measure of the set of times at which there is a component of size at least $A_n n^{2/3}$ converges in probability to zero whenever $A_n \to \infty$, so certainly the Lebesgue measure of the set of exceptional times converges in probability to zero for any $\beta > 0$. On the other hand, let $X_\beta(t)$ be 1 when the largest component is larger than $\beta n^{2/3}\log^{1/3} n$, and 0 at other times. For $\delta > 0$, let $N_\beta(\delta)$ be the number of times in the interval $[0,\delta]$ at which $X_\beta(t)$ changes its value. Jonasson and Steif [16, Corollary 1.6] showed that if $\mathbb{P}(|L_n| \geq \beta n^{2/3}\log^{1/3} n) \to 0$ but $\mathbb{P}(N_\beta(1) \geq 1) \to 1$, then $N_\beta(\delta) \to \infty$ in distribution as $n \to \infty$ for any fixed $\delta > 0$. Theorem 1.1 tells us that these conditions hold for $\beta < 2/3^{2/3}$.

The third question appears to be substantially different from Theorem 1.1 and would require a different approach.

For the fourth question, the most interesting case is $\gamma = -1/3$. If we rescale component sizes by $n^{2/3}$ then, based on Aldous' multiplicative coalescent [1], we expect to see something like a multiplicative fragmentation-coalescent process. Rossignol [22] has shown that this is indeed the case.

1.2 Background

Dynamical percolation was introduced by Häggström, Peres, and Steif [12]. Take a graph $G = (V, E)$ and create a dynamical random graph $(G_t,\, t \geq 0)$ as follows. Each edge $e \in E$ is present at time 0 with probability $p \in [0,1]$, independently of all others. Each edge is then rerandomized independently at the times of a rate 1 Poisson process. This model is known as *dynamical bond percolation* on G with parameter p. (Alternatively we may say that each *vertex* is present with probability p and rerandomized at rate 1; this is known as dynamical site percolation.) The model that we investigate in this paper is then simply dynamical bond percolation when G is the complete graph on n vertices and $p = 1/n$.

Erdős-Rényi graphs are probably the simplest models of real-world networks. Several more complex random graph models, such as preferential attachment graphs, have since been introduced in an attempt to more realistically model the features seen in networks such as the world-wide web; see [14] for an overview. Many real-world networks change over time, and we therefore hope that this article will serve as a first step in investigating dynamical models of such networks.

A question of particular interest for infinite graphs is whether there exists a time at which an infinite component appears. Schramm and Steif [23] were able to show that for critical ($p = 1/2$) dynamical site percolation on the triangular lattice, almost surely, there are times in $[0,\infty)$ when an infinite component is present, even though at any fixed time t there is almost surely no infinite component. The times at which an infinite component exists are then known as *exceptional times*. Their proof relied on tools from Fourier analysis, randomized algorithms, and the theory of noise sensitivity of Boolean functions as introduced by Benjamini, Kalai and Schramm [2]. We will see similar methods appearing in our proof, although in each case there will be a non-standard approach required.

Since its introduction roughly 20 years ago, dynamical percolation has been studied intensively [3, 5, 9, 11, 13, 18] in various settings (see also [8, 10, 24] and references within). Most of the study has so far been restricted to infinite graphs and the question of existence of an infinite component. Of the very few results on finite graphs, Lubetzky and Steif [17] studied the noise sensitivity properties of various Boolean functions related to Erdős-Rényi graphs; and Jonasson and Steif [16] gained results about dynamical percolation on infinite spherically symmetric trees restricted to the first n levels, in the context of what they call the volatility of Boolean functions (which we mentioned using different notation in Section 1.1 in the discussion around the second open question). Rosengren and Trapman [21] considered a slightly different dynamical Erdős-Rényi model, and asked how long it takes to reach a certain number of edges.

1.3 Proof ideas

The proof that if $\beta > 2/3^{1/3}$ then there are no exceptional times (i.e. with high probability there are no times in $[0, 1]$ when there is a component of size bigger than $\beta n^{2/3} \log^{1/3} n$) uses a standard first moment method. We split $[0, 1]$ into many smaller intervals, use known asymptotics for the probability of seeing a large component for p slightly bigger than $1/n$ to bound the probability of seeing an exceptional time on one of these small intervals, and then take a union bound. The main interest of this paper is therefore the result that there are exceptional times for $\beta < 2/3^{2/3}$.

As discussed in [23], in order to see such times, the configuration must "change rapidly" so that it has "many chances" to have a large component. By "change rapidly" we mean that the configurations must have small correlations over short time intervals. To quantify this we use a second moment method, and the key will be to estimate

$$\int_0^1 \int_0^1 \mathbb{P}(|\mathcal{C}_u(s)| > An^{2/3},\, |\mathcal{C}_v(t)| > An^{2/3})\, \mathrm{d}t\, \mathrm{d}s$$

where $A = \beta \log^{1/3} n$ and $\mathcal{C}_v(t)$ is the component containing vertex v at time t.

We will need different methods for estimating $\mathbb{P}(\mathcal{C}_u(0)| > An^{2/3},\, |\mathcal{C}_v(t)| > An^{2/3})$, roughly depending on whether $|t - s|$ is less than or greater than $n^{-2/9}$. For small values of $|t - s|$ we will use a counting argument. For larger values of $|t - s|$ the correlations become harder to control and we will need to use tools from discrete Fourier analysis. A very interesting theory of noise sensitivity has been developed around this concept when $\mathbb{P}$ is a uniform product measure, i.e. when the probability that each edge (or vertex) is present is 1/2: see [10]. Since our measure $\mathbb{P}$ is highly non-symmetric, we must redevelop some of the noise sensitivity tools in our non-standard setting. Even then there are complications and some twists on the theory are needed, which may be of interest in their own right.

The basic idea is to use the notion of randomized algorithms. We aim to design an algorithm which examines some of the edges $e \in E$ (i.e. looks at whether they are present or not), and decides whether or not there is a large component. If for any fixed e, the probability that the algorithm checks e is small, then the Fourier coefficients that we are interested in must also be small. This result is known in the uniform case [23], and the proof carries over to non-uniform $\mathbb{P}$. The major complication here is that we are not able to construct an algorithm with the desired properties, essentially because of the lack of geometry in the graph. To check whether a particular vertex v is in a large component, we need to examine almost all the edges emanating from v. For each v, we are therefore forced to consider two classes of edges. For those edges e that do not have an endpoint at v, we can use a well-known exploration algorithm and use the lack of geometry to our advantage to bound the probability that e is examined. For the edges that do have an endpoint at v, we use a completely different method inspired by the *spectral sample* introduced in [2]. We bound the relevant Fourier coefficients by looking at the probability that the edge e is pivotal, i.e. that there is a large component when e is present and not if e is absent.

2 Fourier analysis of Boolean functions

In this section we give general results on the Fourier analysis of Boolean functions. Several of the results presented here are known in the case when $\mathbb{P}$ is a uniform product measure; see for example [10]. We also note that Talagrand [25] developed hypercontractivity results in the case we are considering, where $\mathbb{P}$ is a homogeneous but non-uniform product measure. We repeat some of his defintions below.

2.1 Definitions and first results

Let E be a finite set and define $\Omega := \{0,1\}^E$. Let $\mathbb{P} = \mathbb{P}_p$ be a measure on Ω defined by

$$\mathbb{P}(\omega) = p^{\#\{e:\omega(e)=1\}}\,(1-p)^{\#\{i:\omega(e)=0\}}\,.$$

All of our results in this section apply to any finite set E and any $p \in [0,1]$. We refer to the elements of E as bits. Of course we have in our minds the application where E is the edge set of the complete graph K_n and $p = 1/n$; and where for $\omega \in \Omega$ we say that edge e is present if and only if $\omega(e) = 1$.

For $\omega \in \Omega$ and $e \in E$ let

$$r_e(\omega) := \begin{cases} \sqrt{\frac{1-p}{p}} & \text{if } \omega(e) = 1 \\ -\sqrt{\frac{p}{1-p}} & \text{if } \omega(e) = 0. \end{cases}$$

For $S \subset E$ let

$$\chi_S(\omega) = \prod_{e \in S} r_e(\omega)$$

where we set $\chi_\emptyset \equiv 1$. Then for any function $f : \Omega \to \mathbb{R}$ and $S \subset E$ we define

$$\hat{f}(S) = \mathbb{E}[f\chi_S],$$

and call $\hat{f}(S)$, for $S \subset E$, the *Fourier coefficients* of f.

It is easy to check that $\{\chi_S : S \subset E\}$ forms an orthonormal basis for $L^2(\mathbb{P})$, and therefore—just as in continuous Fourier analysis—the function $\hat{f}$ encodes all of the information about f, in that $f(\omega) = \sum_{S \subset E} \hat{f}(S)\chi_S(\omega)$.

One simple consequence of the definition is that $\mathbb{E}[f] = \hat{f}(\emptyset)$. Another useful result is *Plancherel's identity*, which states that for two functions $f, g : \Omega \to \mathbb{R}$,

$$\sum_S \hat{f}(S)\hat{g}(S) = \mathbb{E}[fg]. \tag{1}$$

This is easy to prove simply by writing out $f = \sum_S \hat{f}(S)\chi_S$ and $g = \sum_{S'} \hat{g}(S')\chi_{S'}$ and using orthonormality.

Recall from the introduction that we will be interested in bounding probabilities like $\mathbb{P}(|\mathcal{C}_u(s)| > An^{2/3}, |\mathcal{C}_v(t)| > An^{2/3})$, where $\mathcal{C}_v(t)$ is the component containing v at time t in the dynamical Erdős-Rényi graph. We will therefore be appling our Fourier analysis to functions of the form $\mathbb{1}_{\{\mathcal{C}_v(t) > An^{2/3}\}}$. The following lemma, which is already known (see [10, (4.2)] and [2, (2.2)]), will be very useful for this purpose. Given $\omega \in \Omega$ and $\varepsilon \in [0,1]$, let ω_ε be the configuration obtained by rerandomizing each of the bits in ω independently with probability ε. That is, for each $e \in E$,

$$\omega_\varepsilon(e) = \omega(e)\mathbb{1}_{\{U_e > \varepsilon\}} + \mathbb{1}_{\{V_e < p\}}\mathbb{1}_{\{U_e \le \varepsilon\}}$$

where U_e and V_e are independent uniform random variables on $(0,1)$.

Lemma 2.1. *For any $\varepsilon \in [0,1]$ and any $f, g : \Omega \to \mathbb{R}$,*

$$\mathbb{E}[f(\omega)g(\omega_\varepsilon)] = \sum_S \hat{f}(S)\hat{g}(S)(1-\varepsilon)^{|S|}$$

(where the expectation $\mathbb{E}$ averages both over $\omega \in \Omega$ and also over the randomness in the resampling required to create ω_ε).

Proof. Note that

$$\mathbb{E}[f(\omega)g(\omega_\varepsilon)] = \mathbb{E}\Big[\sum_S \hat{f}(S)\chi_S(\omega)\sum_{S'} \hat{g}(S')\chi_{S'}(\omega_\varepsilon)\Big] = \sum_{S,S'} \hat{f}(S)\hat{g}(S')\mathbb{E}[\chi_S(\omega)\chi_{S'}(\omega_\varepsilon)].$$

It is easy to check that if $S \neq S'$ then $\mathbb{E}[\chi_S(\omega)\chi_{S'}(\omega_\varepsilon)] = 0$, and on the other hand that

$$\mathbb{E}[\chi_S(\omega)\chi_S(\omega_\varepsilon)] = \prod_{e\in S}\mathbb{E}[r_e(\omega)r_e(\omega_\varepsilon)] = (1-\varepsilon)^{|S|}.$$

□

2.2 Randomized algorithms and revealment

Evaluating Fourier coefficients directly is often quite difficult and instead we concentrate on bounding sums such as the one on the right-hand side of Lemma 2.1. One approach that has proven fruitful in the past is to introduce a *randomized revealment algorithm* that attempts to decide the value of the function f by revealing $\omega(e)$ only for relatively few of the possible bits $e \in E$. If for any fixed e, the probability that the algorithm reveals $\omega(e)$ is small, then it turns out that the sum of the Fourier coefficients must be small [23, Theorem 1.8]. Our main result in this section is a generalization of [23, Theorem 1.8].

Let $f : \Omega = \{0,1\}^E \to \mathbb{R}$. A revealment algorithm, A, for f is a sequence of bits $e_1, e_2, \ldots, e_T \in E$, chosen one by one, with the choice of e_k possibly depending on the values of $\omega(e_1), \ldots, \omega(e_{k-1})$, and such that knowledge of $\omega(e_1), \ldots, \omega(e_T)$ determines the value of $f(w)$. A *randomized revealment algorithm* is a revealment algorithm that is also allowed to use auxiliary randomness in making choices. Given such an algorithm A, let J be the set of bits revealed by A.

For $U \subset E$, define the revealment of the algorithm A on U by

$$\mathcal{R}_U = \mathcal{R}_U(f, A) := \max_{e \in U^c} \mathbb{P}(e \in J).$$

Our main result in this section is the following generalization of [23, Theorem 1.8].

Theorem 2.2. *Let A be an algorithm determining $f : \Omega \to \mathbb{R}$ and let $U \subset E$. Then for any $k \in \mathbb{N}$,*

$$\sum_{\substack{|S|=k, \\ S \cap U = \emptyset}} \hat{f}(S)^2 \leq \mathcal{R}_U(f, A)\mathbb{E}[f(\omega)^2]k.$$

The result in [23], besides being stated for the uniform measure (i.e. $p = 1/2$), only included the case $U = \emptyset$. The reason that we need a generalization involves the geometry of the Erdős-Rényi graph. As far as we can tell, any algorithm to check whether there is an unusually large component must reveal almost all of the edges emanating from many of the vertices; similarly, any algorithm to check whether a particular vertex v is in an unusually large component must reveal almost all of the edges with an endpoint at v.

To get around this problem we fix a vertex v and separate subsets S of edges into those which contain an edge with an endpoint at v, and those which do not. We then use Theorem 2.2 to bound the Fourier coefficients of the latter sets, and take a different approach to the former. This different approach was inspired by the spectral sample introduced in [2], and will be carried out in Section 2.3.

For now we aim to prove Theorem 2.2. Our strategy is very much based on the proof in [23].

Let $\tau \subset \mathcal{T}$ represent the auxiliary randomness used by the algorithm, and let $\tilde{\mathbb{P}}$ be the canonical probability measure on the extended space $\Omega \times \mathcal{T}$. Let $\mathcal{A}$ be the smallest σ-algebra such that J and $\{\omega(e) : e \in J\}$ are measurable. Note that since A determines the value of f, and $\mathcal{A}$ contains all the information revealed by A, f is $\mathcal{A}$-measurable.

For any function $h : \Omega \to \mathbb{R}$ and $(\omega, \tau) \in \Omega \times \mathcal{T}$, define $h_{J(\omega,\tau)}$ by

$$\begin{aligned} h_{J(\omega,\tau)} : \Omega &\to \mathbb{R} \\ \omega' &\mapsto h(\omega'_{J(\omega,\tau)}), \end{aligned}$$

where

$$\omega'_{J(\omega,\tau)}(e) = \begin{cases} \omega(e) & \text{if } e \in J(\omega, \tau) \\ \omega'(e) & \text{if } e \notin J(\omega, \tau). \end{cases}$$

We now want to be able to take expectations over $\omega' \in \Omega$, using our usual probability measure under which each bit of ω' is 1 with probability p and 0 with probability $1-p$, while keeping ω and τ fixed. We write $\mathbb{P}^{\omega,\tau}$ to emphasise that ω and τ are fixed. The notation $\widehat{h_{J(\omega,\tau)}}(S)$ will mean the Fourier coefficient with respect to $\mathbb{P}^{\omega,\tau}$, i.e. $\mathbb{E}^{\omega,\tau}[h_{J(\omega,\tau)}(\omega')\chi_S(\omega')]$.

We start with a general lemma about any such function h, before choosing a particular h. We stress that these proofs are almost identical to those in [23], but fleshed out and adapted to our more general situation.

Lemma 2.3. *For any $S \subset E$ and any function $h : \Omega \to \mathbb{R}$,*

$$\tilde{\mathbb{E}}[h(\omega)|\mathcal{A}] = \widehat{h_{J(\omega,\tau)}}(\emptyset).$$

Proof. Setting ω^S to be 1 on S and 0 off S, we have

$$\begin{aligned}\widehat{h_{J(\omega,\tau)}}(\emptyset) &= \mathbb{E}^{\omega,\tau}[h_{J(\omega,\tau)}(\omega')] \\ &= \sum_{S\subset E} h_{J(\omega,\tau)}(\omega^S)p^{|S|}(1-p)^{|E\setminus S|} \\ &= \sum_{S\subset E} h(\omega^S_{J(\omega,\tau)})p^{|S|}(1-p)^{|E\setminus S|} \\ &= \sum_{S\subset J(\omega,\tau)^c} h(\omega^{S\cup J'(\omega,\tau)})p^{|S|}(1-p)^{|J(\omega,\tau)^c\setminus S|}\end{aligned}$$

where $J'(\omega,\tau) = \{e \in J(\omega,\tau) : \omega(e) = 1\}$. But this last quantity is exactly $\tilde{\mathbb{E}}[h(\omega)|\mathcal{A}]$. $\square$

We now fix a function h by setting

$$h(\omega) = \sum_{\substack{|S|=k,\\ S\cap U=\emptyset}} \hat{f}(S)\chi_S(\omega).$$

Lemma 2.4. *For any $S \subset E$ with $|S| = k$,*

$$\widehat{h_{J(\omega,\tau)}}(S) = \begin{cases} 0 & \text{if } S\cap J(\omega,\tau) \neq \emptyset \\ \hat{h}(S) & \text{if } S\cap J(\omega,\tau) = \emptyset. \end{cases}$$

Proof. Note that $h_{J(\omega,\tau)}(\omega') = h(\omega'_{J(\omega,\tau)}) = \sum_S \hat{h}(S)\chi_S(\omega'_{J(\omega,\tau)})$. Therefore

$$\widehat{h_{J(\omega,\tau)}}(S) = \mathbb{E}^{\omega,\tau}[h_{J(\omega,\tau)}(\omega')\chi_S(\omega')] = \sum_{|S'|=k} \hat{h}(S')\mathbb{E}^{\omega,\tau}[\chi_{S'}(\omega'_{J(\omega,\tau)})\chi_S(\omega')].$$

If $S' \neq S$, then (since S' and S have the same size) we may take $e \in S \setminus S'$; changing the value of the bit e changes χ_S but not $\chi_{S'}$, so an easy calculation shows that in this case $\mathbb{E}^{\omega,\tau}[\chi_{S'}(\omega'_{J(\omega,\tau)})\chi_S(\omega')] = 0$. Thus

$$\widehat{h_{J(\omega,\tau)}}(S) = \mathbb{E}^{\omega,\tau}[h_{J(\omega,\tau)}(\omega')\chi_S(\omega')] = \hat{h}(S)\mathbb{E}^{\omega,\tau}[\chi_S(\omega'_{J(\omega,\tau)})\chi_S(\omega')].$$

Now if $S \cap J(\omega,\tau) \neq \emptyset$, then we may take $e \in S \cap J(\omega,\tau)$; since $e \in J(\omega,\tau)$, the value of $\omega'_{J(\omega,\tau)}$ remains constant when we change $\omega'(e)$. On the other hand, since $e \in S$, the value of $\chi_S(\omega')$ changes when we change $\omega'(e)$. Therefore another easy calculation gives that in this case also $\mathbb{E}^{\omega,\tau}[\chi_S(\omega'_{J(\omega,\tau)})\chi_S(\omega')] = 0$, and thus $\widehat{h_{J(\omega,\tau)}}(S) = 0$ when $S \cap J(\omega,\tau) \neq \emptyset$.

Finally, if $S \cap J(\omega,\tau) = \emptyset$, then $\chi_S(\omega'_{J(\omega,\tau)}) = \chi_S(\omega')$, so in this case by orthonormality $\mathbb{E}^{\omega,\tau}[\chi_S(\omega'_{J(\omega,\tau)})\chi_S(\omega')] = 1$ and $\widehat{h_{J(\omega,\tau)}}(S) = \hat{h}(S)$. This completes the proof. $\square$

Lemma 2.5. *We have*

$$\tilde{\mathbb{E}}[\widehat{h_{J(\omega,\tau)}}(\emptyset)^2] \le \sum_{\substack{|S|=k \\ S\cap U=\emptyset}} \hat{h}(S)^2 \tilde{\mathbb{P}}(J \cap S \neq \emptyset).$$

Proof. Using Plancherel's identity on the function $h_{J(\omega,\tau)}$, we have

$$\mathbb{E}^{\omega,\tau}[h_{J(\omega,\tau)}(\omega')^2] = \sum_S \widehat{h_{J(\omega,\tau)}}(S)^2$$

and therefore

$$\widehat{h_{J(\omega,\tau)}}(\emptyset)^2 = \mathbb{E}^{\omega,\tau}[h_{J(\omega,\tau)}(\omega')^2] - \sum_{|S|>0} \widehat{h_{J(\omega,\tau)}}(S)^2. \tag{2}$$

If we let $g = h^2$, then applying Lemma 2.3 to g and using Plancherel's identity we see that

$$\tilde{\mathbb{E}}[\mathbb{E}^{\omega,\tau}[h_{J(\omega,\tau)}(\omega')^2]] = \tilde{\mathbb{E}}[\widehat{g_{J(\omega,\tau)}}(\emptyset)] = \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[g(\omega)|\mathcal{A}]] = \tilde{\mathbb{E}}[g(\omega)] = \tilde{\mathbb{E}}[h(\omega)^2] = \sum_S \hat{h}(S)^2.$$

Therefore, taking expectations in (2), we get

$$\tilde{\mathbb{E}}[\widehat{h_{J(\omega,\tau)}}(\emptyset)^2] = \sum_S \hat{h}(S)^2 - \sum_{|S|>0} \tilde{\mathbb{E}}[\widehat{h_{J(\omega,\tau)}}(S)^2].$$

By Lemma 2.4, $\widehat{h_{J(\omega,\tau)}}(S)^2 = \hat{h}(S)^2 \mathbb{1}_{\{J\cap S=\emptyset\}}$ when $|S| = k$; and the same quantity is obviously non-negative when $|S| \neq k$, so

$$\tilde{\mathbb{E}}[\widehat{h_{J(\omega,\tau)}}(\emptyset)^2] \le \sum_S \hat{h}(S)^2 - \sum_{|S|=k} \hat{h}(S)^2 \tilde{\mathbb{P}}(J \cap S = \emptyset).$$

Since $\hat{h}(S) = 0$ unless $|S| = k$ and $S \cap U = \emptyset$, the result follows. $\square$

We can now prove Theorem 2.2.

Proof of Theorem 2.2. We claim first that

$$\tilde{\mathbb{E}}[h(\omega)^2]^2 \le \tilde{\mathbb{E}}[f(\omega)^2]\tilde{\mathbb{E}}[\widehat{h_{J(\omega,\tau)}}(\emptyset)^2]. \tag{3}$$

To show this, note that by orthogonality,

$$\begin{aligned}\tilde{\mathbb{E}}[h(\omega)f(\omega)] &= \tilde{\mathbb{E}}\Big[\sum_{\substack{|S|=k; \\ S\cap U=\emptyset}} \hat{f}(S)\chi_S(\omega) \sum_{S'\subset E} \hat{f}(S')\chi_{S'}(\omega)\Big] \\ &= \tilde{\mathbb{E}}\Big[\sum_{\substack{|S|=k; \\ S\cap U=\emptyset}} \hat{f}(S)\chi_S(\omega) \sum_{\substack{|S'|=k; \\ S'\cap U=\emptyset}} \hat{f}(S')\chi_{S'}(\omega)\Big] = \tilde{\mathbb{E}}[h(\omega)^2].\end{aligned}$$

On the other hand,

$$\tilde{\mathbb{E}}[h(\omega)f(\omega)] = \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[h(\omega)f(\omega)|\mathcal{A}]] = \tilde{\mathbb{E}}[f(\omega)\tilde{\mathbb{E}}[h(\omega)|\mathcal{A}]] \leq \tilde{\mathbb{E}}[f(\omega)^2]^{1/2}\tilde{\mathbb{E}}[\tilde{\mathbb{E}}[h(\omega)|\mathcal{A}]^2]^{1/2}$$

where the second equality uses the fact that f is $\mathcal{A}$-measurable, and the last inequality uses Cauchy-Schwartz. Putting these two expressions for $\tilde{\mathbb{E}}[h(\omega)f(\omega)]$ together, and recalling from Lemma 2.3 that $\tilde{\mathbb{E}}[h(\omega)|\mathcal{A}] = \widehat{h_{J(\omega,\tau)}}(\emptyset)$, we get (3).

Now, combining (3) with Lemma 2.5,

$$\tilde{\mathbb{E}}[h(\omega)^2]^2 \leq \tilde{\mathbb{E}}[f(\omega)^2] \sum_{\substack{|S|=k,\\ S\cap U=\emptyset}} \hat{h}(S)^2\tilde{\mathbb{P}}(J\cap S \neq \emptyset).$$

Taking a union bound, for any S with $|S| = k$ and $S \cap U = \emptyset$ we have $\tilde{\mathbb{P}}(J \cap S \neq \emptyset) \leq k\mathcal{R}_U$, so

$$\tilde{\mathbb{E}}[h(\omega)^2]^2 \leq \tilde{\mathbb{E}}[f(\omega)^2] \sum_{\substack{|S|=k;\\ S\cap U=\emptyset}} \hat{h}(S)^2 k\mathcal{R}_U.$$

By Plancherel's identity and the definition of h,

$$\sum_{\substack{|S|=k;\\ S\cap U=\emptyset}} \hat{h}(S)^2 = \sum_{S} \hat{h}(S)^2 = \tilde{\mathbb{E}}[h(\omega)^2], \tag{4}$$

so

$$\tilde{\mathbb{E}}[h(\omega)^2]^2 \leq \tilde{\mathbb{E}}[f(\omega)^2]\tilde{\mathbb{E}}[h(\omega)^2]k\mathcal{R}_U$$

and therefore $\tilde{\mathbb{E}}[h(\omega)^2] \leq \tilde{\mathbb{E}}[f(\omega)^2]k\mathcal{R}_U$. Since $\hat{h}(S) = \hat{f}(S)$ for all S with $|S| = k$ and $S \cap U = \emptyset$, using (4) again we have

$$\sum_{\substack{|S|=k;\\ S\cap U=\emptyset}} \hat{f}(S)^2 = \tilde{\mathbb{E}}[h(\omega)^2] \leq \tilde{\mathbb{E}}[f(\omega)^2]k\mathcal{R}_U. \qquad \square$$

2.3 Pivotality

In Section 2.2 we gave a method for bounding

$$\sum_{\substack{|S|=k,\\ S\cap U=\emptyset}} \hat{f}(S)^2,$$

which we will apply by fixing a vertex v and letting U be the set of edges that do not have an endpoint at v. In this section we will give a bound on the Fourier coefficients of sets that *do* contain a particular edge, using the notion of pivotality.

An edge $e \in E$ is said to be *pivotal* for f and $\omega \in \Omega$ if $f(\sigma_e(\omega)) \neq f(\omega)$, where $\sigma_e(\omega)$ is the configuration obtained from ω by switching the value of $\omega(e)$. Let $\mathcal{P}_f = \mathcal{P}_f(\omega)$ denote the set of pivotal edges. The next lemma allows us to control the Fourier coefficients by estimating the probability of being pivotal. Similar results are known in the case when $\mathbb{P}$ is a uniform measure; see [10, Proposition 4.4 and Chapter 9]. The non-uniform case is somewhat more delicate.

Lemma 2.6. *Let $f, g : \Omega \to \{0,1\}$. Then for any $e \in E$,*

$$\sum_{S: e \in S} \hat{f}(S)\hat{g}(S) = p(1-p)\mathbb{P}(e \in \mathcal{P}_f \cap \mathcal{P}_g).$$

Proof. Fix $e \in E$ and define an operator ∇_e by setting

$$\nabla_e f(\omega) = |r_e(\omega)|\Big(f(\omega) - f(\sigma_e(\omega))\Big).$$

Since $f(\omega) = \sum_S \hat{f}(S)\chi_S(\omega)$, from the definition of χ_S we have that

$$\nabla_e f(\omega) = |r_e(\omega)|\Big(r_e(\omega) - r_e(\sigma_e(\omega))\Big) \sum_{S: e \in S} \hat{f}(S)\chi_{S \setminus \{e\}}(\omega)$$

Now, if $\omega(e) = 1$, then $r_e(\omega) = ((1-p)/p)^{1/2}$ and $r_e(\sigma_e(\omega)) = -(p/(1-p))^{1/2}$ and so

$$\begin{aligned}
|r_e(\omega)|\Big(r_e(\omega) - r_e(\sigma_e(\omega))\Big) &= \left(\frac{1-p}{p}\right)^{1/2} \left(\left(\frac{1-p}{p}\right)^{1/2} + \left(\frac{p}{1-p}\right)^{1/2}\right) \\
&= 1/p \\
&= \frac{r_e(\omega)}{p^{1/2}(1-p)^{1/2}}.
\end{aligned}$$

On the other hand if $\omega(e) = 0$, then $r_e(\omega) = -(p/(1-p))^{1/2}$ and $r_e(\sigma_e(\omega)) = ((1-p)/p)^{1/2}$ so that

$$\begin{aligned}
|r_e(\omega)|\Big(r_e(\omega) - r_e(\sigma_e(\omega))\Big) &= \left(\frac{p}{1-p}\right)^{1/2} \left(-\left(\frac{p}{1-p}\right)^{1/2} - \left(\frac{1-p}{p}\right)^{1/2}\right) \\
&= -1/(1-p) \\
&= \frac{r_e(\omega)}{p^{1/2}(1-p)^{1/2}}.
\end{aligned}$$

Thus either way, we see that

$$\nabla_e f(\omega) = \frac{1}{p^{1/2}(1-p)^{1/2}} \sum_{S: e \in S} \hat{f}(S)\chi_S(\omega).$$

It follows that

$$\widehat{\nabla_e f}(S) = \begin{cases} p^{-1/2}(1-p)^{-1/2}\hat{f}(S) & \text{if } e \in S \\ 0 & \text{if } e \notin S \end{cases}$$

and by Plancherel's identity (1),

$$\mathbb{E}\left[(\nabla_e f)(\nabla_e g)\right] = \sum_S \widehat{\nabla_e f}(S)\widehat{\nabla_e g}(S) = \frac{1}{p(1-p)} \sum_{S: e \in S} \hat{f}(S)\hat{g}(S). \tag{5}$$

Next we compute $\mathbb{E}\left[(\nabla_e f)(\nabla_e g)\right]$ directly. Notice that

$$\nabla_e f(\omega)\nabla_e g(\omega) = \begin{cases} (1-p)/p & \text{if } e \in \mathcal{P}_f(\omega) \cap \mathcal{P}_g(\omega) \text{ and } \omega(e) = 1 \\ p/(1-p) & \text{if } e \in \mathcal{P}_f(\omega) \cap \mathcal{P}_g(\omega) \text{ and } \omega(e) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since the event $\{e \in \mathcal{P}_f; e \in \mathcal{P}_g\}$ is independent of the event $\{\omega(e) = \pm 1\}$ we see that

$$\mathbb{E}\left[(\nabla_e f)(\nabla_e g)\right] = p\frac{1-p}{p}\mathbb{P}(e \in \mathcal{P}_f \cap \mathcal{P}_g) + (1-p)\frac{p}{1-p}\mathbb{P}(e \in \mathcal{P}_f \cap \mathcal{P}_g) = \mathbb{P}(e \in \mathcal{P}_f \cap \mathcal{P}_g).$$

The lemma now follows by combining this with (5). □

3 Component sizes of Erdős-Rényi graphs

In this section we collect some preliminary results about component sizes for Erdős-Rényi graphs, which will be useful later on. We let $\mathbf{P}_{n,p}$ be the law of $\mathrm{ER}(n,p)$. We begin by presenting a result that gives the tail behaviour of the size of components. For a full proof of Proposition 3.1, see [20]. Pittel [19, Proposition 2] proved part (b) when λ is fixed and a is large but does not depend on n.

Proposition 3.1. *Let G be an $ER(n, 1/n-\lambda_n n^{-4/3})$ random graph. Write $p = 1/n-\lambda_n n^{-4/3}$. Suppose that $A_n \to \infty$ as $n \to \infty$, $A_n = o(n^{1/12})$, and $|\lambda_n| = o(A_n)$. Then as $n \to \infty$,*

(a) $\mathbf{P}_{n,p}(|\mathcal{C}_v| \geq A_n n^{2/3}) = \dfrac{e^{-A_n^3/8}}{(9\pi/8)^{1/2} A_n^{1/2} n^{1/3}}(1+o(1))$;

(b) $\mathbf{P}_{n,p}\left(\max_v |\mathcal{C}_v| \geq A_n n^{2/3}\right) = \dfrac{e^{-A_n^3/8}}{(9\pi/8)^{1/2} A_n^{3/2}}(1+o(1))$.

We will also need bounds on $\mathbf{P}_{n,p}(|\mathcal{C}_v| = k)$. Again we refer to [20] for a proof.

Lemma 3.2. *Let $G = (V,E)$ be an $ER(n, 1/n - \lambda_n n^{-4/3})$ random graph. Let $p = 1/n - \lambda_n n^{-4/3}$. There exist constants $0 < c_1 \leq c_2 < \infty$ such that*

(a) if $k \leq n^{2/3}$ and $\lambda_n = o(n^{1/12})$, then for all large n

$$\frac{c_1}{k^{3/2}} \leq \mathbf{P}_{n,p}(|\mathcal{C}_v| = k) \leq \frac{c_2}{k^{3/2}};$$

(b) if $n^{2/3} \leq k \leq n^{3/4}$ and $|\lambda| \leq n^{1/12}$, then for all large n

$$\frac{c_1 k^{3/2}}{n^2} e^{-G_\lambda(k/n^{2/3})} \leq \mathbf{P}_{n,p}(|\mathcal{C}_v| = k) \leq \frac{c_2 k^{3/2}}{n^2} e^{-G_\lambda(k/n^{2/3})}$$

where $G_\lambda(x) = x^3/8 - \lambda x^2/2 + \lambda^2 x/2$.

We give two more lemmas, which follow fairly easily from those above, but are less obviously useful. We will see later that they are exactly the bounds we need to estimate the probability that two vertices have unusually large components at different times.

Lemma 3.3. *There exists a finite constant c such that whenever $n^{2/3} \leq k \leq n^{3/4}$ and $j \geq (n-k)^{2/3}$,*

$$\mathbf{P}_{n-k,1/n}(|\mathcal{C}_v| = j) \leq \frac{cj^{3/2}}{n^2} \exp\Big(- \frac{(k+j)^3}{8n^2} + \frac{k^3}{8n^2}\Big).$$

Proof. We want to apply Lemma 3.2. Note that

$$\frac{1}{n} = \frac{1}{(n-k)(1+k/(n-k))} \leq \frac{1}{n-k}\Big(1 - \frac{k}{n-k} + \frac{k^2}{(n-k)^2}\Big).$$

Therefore, setting $\lambda = -\frac{k}{(n-k)^{2/3}} + \frac{k^2}{(n-k)^{5/3}}$ and $p = 1/(n-k)^{2/3} - \lambda(n-k)^{-4/3}$, we have

$$\mathbf{P}_{n-k,1/n}(|\mathcal{C}_v| = j) \leq \mathbf{P}_{n-k,p}(|\mathcal{C}_3| = j).$$

Applying Lemma 3.2(b), we get

$$\mathbf{P}_{n-k,1/n}(|\mathcal{C}_v| = j) \leq \frac{cj^{3/2}}{(n-k)^2} e^{-G_\lambda(j/(n-k)^{2/3})}.$$

Now we note that, for $j \leq k \leq n$,

$$\begin{aligned}
\frac{k^3}{8n^2} + G_\lambda\Big(\frac{j}{(n-k)^{2/3}}\Big) &= \frac{k^3}{8n^2} + \frac{j^3}{8(n-k)^2} - \frac{\lambda j^2}{2(n-k)^{4/3}} + \frac{\lambda^2 j}{2(n-k)^{2/3}} \\
&= \frac{k^3}{8n^2} + \frac{j^3}{8(n-k)^2} + \frac{kj^2}{2(n-k)^2} + \frac{k^2 j}{2(n-k)^2} + O\Big(\frac{k^4}{n^3}\Big) \\
&\geq \frac{(k+j)^3}{8n^2} + O\Big(\frac{k^4}{n^3}\Big).
\end{aligned}$$

Therefore we have

$$\mathbf{P}_{n-k,1/n}(|\mathcal{C}_v| = j) \leq \frac{cj^{3/2}}{(n-k)^2} \exp\Big(- \frac{(k+j)^3}{8n^2} + \frac{k^3}{8n^2} + O\Big(\frac{k^4}{n^3}\Big)\Big).$$

Since $k \leq n^{3/4}$, the result follows. $\square$

Lemma 3.4. *Then there exists a finite constant c such that whenever $n^{2/3} \ll N \leq n^{3/4}$, for large n we have*

$$\boldsymbol{P}_{n,1/n}\Big(|\mathcal{C}_u \cup \mathcal{C}_v| \geq N,\, |\mathcal{C}_u| < N,\, |\mathcal{C}_v| < N,\, \mathcal{C}_u \cap \mathcal{C}_v = \emptyset\Big) \leq c\Big(\frac{N^5}{n^4} + \frac{N^{3/2}}{n^{5/3}}\Big) e^{-N^3/(8n^2)}.$$

Proof. Clearly

$$
\begin{aligned}
&\mathbf{P}_{n,1/n}\Big(|\mathcal{C}_u \cup \mathcal{C}_v| \geq N,\ |\mathcal{C}_u| < N,\ |\mathcal{C}_v| < N,\ \mathcal{C}_u \cap \mathcal{C}_v = \emptyset\Big) \\
&\qquad \leq 2\mathbf{P}_{n,1/n}\Big(|\mathcal{C}_u \cup \mathcal{C}_v| \geq N,\ |\mathcal{C}_v| \leq |\mathcal{C}_u| < N,\ \mathcal{C}_u \cap \mathcal{C}_v = \emptyset\Big) \\
&\qquad = 2 \sum_{k=\lceil N/2 \rceil}^{N-1} \sum_{j=N-k}^{k} \mathbf{P}_{n,1/n}\Big(|\mathcal{C}_u| = k\Big)\mathbf{P}_{n,1/n}\Big(|\mathcal{C}_u| = j,\ \mathcal{C}_u \cap \mathcal{C}_v = \emptyset \,\Big|\, |\mathcal{C}_v| = k\Big) \\
&\qquad = 2 \sum_{k=\lceil N/2 \rceil}^{N-1} \sum_{j=N-k}^{k} \mathbf{P}_{n,1/n}\Big(|\mathcal{C}_u| = k\Big)\mathbf{P}_{n-k,1/n}\Big(|\mathcal{C}_u| = j\Big).
\end{aligned}
$$

By Lemma 3.2(b), for $n^{2/3} \leq k \leq n^{3/4}$,

$$
\mathbf{P}_{n,1/n}(|\mathcal{C}_u| = k) \leq \frac{ck^{3/2}}{n^2} e^{-k^3/(8n^2)}.
$$

If $j \geq (n-k)^{2/3}$ then by Lemma 3.3,

$$
\mathbf{P}_{n-k,1/n}\Big(|\mathcal{C}_u| = j\Big) \leq \frac{cj^{3/2}}{n^2} \exp\Big(-\frac{(k+j)^3}{8n^2} + \frac{k^3}{8n^2}\Big);
$$

and if $j \leq (n-k)^{2/3}$ then by Lemma 3.2(a),

$$
\mathbf{P}_{n-k,1/n}\Big(|\mathcal{C}_u| = j\Big) \leq \frac{c}{j^{3/2}}.
$$

Putting these estimates together, we get

$$
\begin{aligned}
&\mathbf{P}_{n,1/n}\Big(|\mathcal{C}_u \cup \mathcal{C}_v| \geq N,\ |\mathcal{C}_u| < N,\ |\mathcal{C}_v| < N,\ \mathcal{C}_u \cap \mathcal{C}_v = \emptyset\Big) \\
&\qquad \leq 2 \sum_{k=\lceil N/2 \rceil}^{N-1} \sum_{j=N-k}^{k} \frac{ck^{3/2}}{n^2} e^{-k^3/(8n^2)} \cdot \frac{cj^{3/2}}{n^2} \exp\Big(-\frac{(k+j)^3}{8n^2} + \frac{k^3}{8n^2}\Big) \\
&\qquad\qquad + 2 \sum_{k=\lceil N - n^{2/3} \rceil}^{N-1} \sum_{j=N-k}^{\lfloor (n-k)^{2/3} \rfloor} \frac{ck^{3/2}}{n^2} e^{-k^3/(8n^2)} \cdot \frac{c}{j^{3/2}}.
\end{aligned}
$$

It is easy to check that

$$
\sum_{k=\lceil N/2 \rceil}^{N-1} \sum_{j=N-k}^{k} \frac{ck^{3/2}}{n^2} e^{-k^3/(8n^2)} \cdot \frac{cj^{3/2}}{n^2} \exp\Big(-\frac{(k+j)^3}{8n^2} + \frac{k^3}{8n^2}\Big) \leq c' \frac{N^5}{n^4} e^{-N^3/(8n^2)},
$$

and (using the fact that $N \gg n^{2/3}$)

$$
\sum_{k=\lceil N - n^{2/3} \rceil}^{N-1} \sum_{j=N-k}^{\lfloor (n-k)^{2/3} \rfloor} \frac{ck^{3/2}}{n^2} e^{-k^3/(8n^2)} \cdot \frac{c}{j^{3/2}} \leq c' \frac{N^{3/2}}{n^{5/3}} e^{-N^3/(8n^2)}.
$$

This completes the proof. $\square$

4 Exceptional times exist for $\beta < 2/3$

In this section we aim to show that if $\beta < 2/3$, then with high probability there exist times $t \in [0,1]$ when $|L_n(t)| > \beta n^{2/3} \log^{1/3} n$. Let $I = [\beta n^{2/3}\log^{1/3} n, 2\beta n^{2/3}\log^{1/3} n] \cap \mathbb{N}$ and for $v \in \{1,\dots,n\}$ let

$$Z_v := \int_0^1 \mathbb{1}_{\{|\mathcal{C}_v(t)| \in I\}} \mathrm{d}t.$$

Then by Cauchy-Schwarz and symmetry we have that

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,1]} |L_n(t)| \geq \beta n^{2/3}\log^{1/3} n\right) &\geq \mathbb{P}\left(\sum_{v=1}^n Z_v > 0\right) \\ &\geq \frac{\mathbb{E}\left[\sum_{v=1}^n Z_v\right]^2}{\mathbb{E}\left[\left(\sum_{v=1}^n Z_v\right)^2\right]} \\ &= \frac{n^2\mathbb{E}[Z_1]^2}{n\mathbb{E}[Z_1^2] + n(n-1)\mathbb{E}[Z_1 Z_2]}. \end{aligned} \tag{6}$$

We begin with a lemma which ensures that the term $n\mathbb{E}[Z_1^2]$ in the denominator of (6) does not contribute substantially when β is small.

Lemma 4.1. *If $\beta^3 < 16/3$, then*

$$\lim_{n\to\infty} \frac{\mathbb{E}[Z_1^2]}{n\mathbb{E}[Z_1]^2} = 0.$$

Proof. By Fubini's theorem, the stationarity in distribution of $\mathcal{C}_1(t)$, and Proposition 3.1,

$$\mathbb{E}[Z_1] = \int_0^1 \mathbb{P}(|\mathcal{C}_1(t)| \in I)\mathrm{d}t = \mathbb{P}(|\mathcal{C}_1(0))| \in I) = (1+o(1))\frac{n^{-\beta^3/8-1/3}}{((9\pi/8)\beta\log n)^{1/2}}. \tag{7}$$

Clearly $Z_1 \leq 1$ so $\mathbb{E}[Z_1^2] \leq \mathbb{E}[Z_1]$, so by (7),

$$\frac{\mathbb{E}[Z_1^2]}{n\mathbb{E}[Z_1]^2} \leq \frac{1}{n\mathbb{E}[Z_1]} \leq Cn^{\beta^3/8-2/3}\beta^{1/2}\log^{1/2} n$$

for some constant C. The lemma follows. ☐

Now using Lemma 4.1 with (6), it remains to show that

$$\limsup_{n\to\infty} \frac{\mathbb{E}[Z_1 Z_2]}{\mathbb{E}[Z_1]^2} \leq 1.$$

Notice that by Fubini's theorem,

$$\mathbb{E}[Z_1Z_2] = \int_0^1\int_0^1 \mathbb{P}(|\mathcal{C}_1(s))| \in I; |\mathcal{C}_2(t)| \in I)\,\mathrm{d}t\,\mathrm{d}s. \tag{8}$$

We will estimate the double integral on the right hand side of (8) by splitting it into two pieces. We begin with an estimate for when $|t-s|$ is small.

4.1 Small $|t-s|$: a combinatorial method

Lemma 4.2. *Let $P = \mathbb{P}(|\mathcal{C}_v| \geq \beta n^{2/3} \log^{1/3} n)$. Then for any $\delta > 0$,*

$$\int_0^1 \int_0^1 \mathbb{P}(|\mathcal{C}_1(s)| \in I; |\mathcal{C}_2(t)| \in I) \mathbb{1}_{\{|t-s| \leq \delta\}} \, dt \, ds \leq 2\delta P^2 + \frac{4\beta \log^{1/3} n}{n^{1/3}} \delta P + 2\delta^2 P.$$

Proof. First note that, by stationarity,

$$\int_0^1 \int_0^1 \mathbb{P}(|\mathcal{C}_1(s)| \in I; |\mathcal{C}_2(t)| \in I) \mathbb{1}_{\{|t-s| \leq \delta\}} \, dt \, ds \leq 2 \int_0^\delta \mathbb{P}(|\mathcal{C}_1(0)| \in I; |\mathcal{C}_2(t)| \in I) \, dt. \tag{9}$$

Now fix $\delta \in [0,1]$ and let $t \in [0, \delta]$. We partition $\mathbb{P}(|\mathcal{C}_1(0)| \in I; |\mathcal{C}_2(t)| \in I)$ into three cases and analyse each case separately. Recall that $\mathbf{P}_{n,p}$ denotes the law of an Erdős-Rényi graph $\mathrm{ER}(n,p)$.

First consider the case when $|\mathcal{C}_1(0) \cap \mathcal{C}_2(t)| = 0$. Then

$$\begin{aligned}
&\mathbb{P}(|\mathcal{C}_1(0)| \in I; |\mathcal{C}_2(t)| \in I; |\mathcal{C}_1(0) \cap \mathcal{C}_2(t)| = 0) \\
&= \sum_{k \in I} \mathbb{P}\Big(|\mathcal{C}_1(0)| = k; |\mathcal{C}_1(0) \cap \mathcal{C}_2(t)| = 0\Big) \mathbb{P}\Big(|\mathcal{C}_2(t)| \in I \,\Big|\, |\mathcal{C}_1(0)| = k; |\mathcal{C}_1(0) \cap \mathcal{C}_2(t)| = 0\Big) \\
&\leq \sum_{k \in I} \mathbf{P}_{n,1/n}(|\mathcal{C}_1| = k) \mathbf{P}_{n-k,1/n}(|\mathcal{C}_2| \in I) \\
&\leq \sum_{k \geq n^{2/3} \log^\beta n} \mathbf{P}_{n,1/n}(|\mathcal{C}_1| = k) \mathbf{P}_{n-k,1/n}(|\mathcal{C}_2| \geq \beta n^{2/3} \log^{1/3} n) \\
&\leq P^2
\end{aligned} \tag{10}$$

where in the final inequality we have used the monotonicity of the event $\{|\mathcal{C}_2| \geq \beta n^{2/3} \log^{1/3} n\}$ in the number of vertices of the graph.

The second case that we look at is when $2 \in \mathcal{C}_1(0)$. In this case

$$\begin{aligned}
\mathbb{P}\Big(|\mathcal{C}_1(0)| \in I; |\mathcal{C}_2(t)| \in I; 2 \in \mathcal{C}_1(0)\Big) &\leq \mathbb{P}\Big(|\mathcal{C}_1(0)| \in I\Big) \mathbb{P}\Big(2 \in \mathcal{C}_1(0) \,\Big|\, |\mathcal{C}_1(0)| \in I\Big) \\
&\leq P \frac{2\beta n^{2/3} \log^{1/3} n}{n}.
\end{aligned} \tag{11}$$

Finally we are left to estimate the probability of the event

$$\{|\mathcal{C}_1(0)| \in I; |\mathcal{C}_2(t)| \in I; |\mathcal{C}_1(0) \cap \mathcal{C}_2(t)| > 0; 2 \notin \mathcal{C}_1(0)\}.$$

Take $A \subset \{1, \ldots, n\}$ such that $|A| \in I$, and condition on $\mathcal{C}_1(0) = A$. Since $|\mathcal{C}_1(0) \cap \mathcal{C}_2(t)| > 0$ and $2 \notin \mathcal{C}_1(0)$, there must be a path π such that

(i) π starts at a vertex $v \in A$ and ends at the vertex 2,

(ii) π first crosses an edge connecting A to A^c, and otherwise only uses edges with both end points in A^c,

(iii) all of the edges in π are open at time t.

We now estimate the probability of such a path existing.

There are at most $2\beta n^{2/3}\log^{1/3} n$ vertices in A, and at most $n-\beta n^{2/3}\log^{1/3} n$ vertices in A^c, so the number of paths of length k satisfying (i) and (ii) is at most $(2\beta n^{2/3}\log^{1/3} n)(n-\beta n^{2/3}\log^{1/3} n)^{k-1}$. Under the conditioning $\mathcal{C}_1(0)=A$, every edge e with both end points lying in A^c is open at time t with probability $1/n$. Moreover, any edge e' with one end point in A and the other in A^c is open at time t with probability $(1-e^{-t})/n$: we know that at time 0 the edge e' is closed (since A is not connected to A^c), and thus in order for it to be open at time t we must first resample the edge, and then open the edge at the resampling. Thus in conclusion we see that the probability there exists a path π of length k satisfying (i), (ii) and (iii) is at most

$$(2\beta n^{2/3}\log^{1/3} n)(n-\beta n^{2/3}\log^{1/3} n)^{k-1}\cdot\frac{1}{n^{k-1}}\cdot\frac{1-e^{-t}}{n} \\ \leq 2t(1-\beta n^{-1/3}\log^{1/3} n)^{k-1}\beta n^{-1/3}\log^{1/3} n$$

where for the inequality we have used the fact that $1-e^{-t}\leq t$. Summing over k, we see that the probability there exists a path π satisfying (i), (ii) and (iii) is at most

$$2t\beta n^{-1/3}\log^{1/3} n\sum_{k=1}^{\infty}(1-\beta n^{-1/3}\log^{1/3} n)^{k-1}=2t.$$

Hence we obtain

$$\mathbb{P}(|\mathcal{C}_1(0)|\in I;|\mathcal{C}_2(t)|\in I;|\mathcal{C}_1(0)\cap\mathcal{C}_2(t)|>0;2\notin\mathcal{C}_1(0))\leq 2t. \tag{12}$$

Putting together (10), (11) and (12) we get

$$\mathbb{P}(|\mathcal{C}_1(0)|\in I;|\mathcal{C}_2(t)|\in I)\leq P^2+P\frac{2\beta n^{2/3}\log^{1/3} n}{n}+2t.$$

Integrating over $t\in[0,\delta]$ and using (9) gives the desired result. □

4.2 Large $|t-s|$: applying Fourier analysis

Fix $\delta>0$. Our next aim is to estimate the integral

$$\int_0^1\int_0^1\mathbb{P}(|\mathcal{C}_1(s)|\in I;|\mathcal{C}_2(t)|\in I)\mathbb{1}_{\{|t-s|>\delta\}}\,\mathrm{d}t\,\mathrm{d}s.$$

To do this, we will use the Fourier analysis introduced in Section 2.

For $v\in\{1,\dots,n\}$, let $f_v:\Omega\to\{0,1\}$ be the function given by

$$f_v(\omega)=\begin{cases}1 & \text{if the connected component of } v \text{ in } \omega \text{ has size lying in the interval } I\\ 0 & \text{otherwise.}\end{cases}$$

We recall some notation from Section 2. For $\omega \in \Omega$ and $\varepsilon \in [0,1]$, let ω_ε be the random configuration obtained from ω by resampling each edge in ω with probability ε. Lemma 2.1 told us that

$$\mathbb{E}[f_1(\omega) f_2(\omega_\varepsilon)] = \sum_S \hat{f}_1(S)\hat{f}_2(S)(1-\varepsilon)^{|S|}. \tag{13}$$

In our setting of the dynamical Erdős-Rényi graph, the configuration at time $t > s$ can be obtained from the configuration at time s by resampling each edge with probability $\varepsilon = 1 - e^{-(t-s)}$. Hence for any $\delta \in (0,1)$,

$$\begin{aligned}
\int_0^1 \int_0^1 \mathbb{P}(|\mathcal{C}_1(s)| \in I; |\mathcal{C}_2(t)| \in I) \mathbb{1}_{\{|t-s|>\delta\}} \,\mathrm{d}t\,\mathrm{d}s &= \int_0^1 \int_0^1 \mathbb{E}[f_1(\omega) f_2(\omega_{1-e^{-|t-s|}})] \mathbb{1}_{\{|t-s|>\delta\}} \,\mathrm{d}t\,\mathrm{d}s \\
&= \sum_S \hat{f}_1(S)\hat{f}_2(S) \int_0^1 \int_0^1 e^{-|t-s||S|} \mathbb{1}_{\{|t-s|>\delta\}} \,\mathrm{d}t\,\mathrm{d}s \\
&\le \hat{f}_1(\emptyset)\hat{f}_2(\emptyset) + \sum_{|S|>0} \hat{f}_1(S)\hat{f}_2(S) \cdot 2\int_\delta^1 e^{-t|S|}\,\mathrm{d}t \\
&= \hat{f}_1(\emptyset)\hat{f}_2(\emptyset) + 2\sum_{|S|>0} \frac{e^{-\delta|S|}}{|S|} \hat{f}_1(S)\hat{f}_2(S) \\
&= \mathbb{E}[Z_1]^2 + 2\sum_{|S|>0} \frac{e^{-\delta|S|}}{|S|} \hat{f}_1(S)\hat{f}_2(S)
\end{aligned} \tag{14}$$

where in the final equality we have used the fact that $\hat{f}_v(\emptyset) = \mathbb{P}(|\mathcal{C}_v(0)| \in I) = \mathbb{E}[Z_1]$.

Let $\mathcal{U}_v$ be the set of edges that have an end point at v. We will study the Fourier coefficients $\hat{f}_1(S)\hat{f}_2(S)$ by separating into cases when $S \cap (\mathcal{U}_1 \cup \mathcal{U}_2) \neq \emptyset$ and when $S \cap (\mathcal{U}_1 \cup \mathcal{U}_2) = \emptyset$. For the former case we will apply Lemma 2.6, and for the latter we will use Theorem 2.2. We begin by studying the former.

Lemma 4.3. *There exists a finite constant C such that for all large n,*

$$\sum_{S: S\cap(\mathcal{U}_1\cup\mathcal{U}_2)\neq\emptyset} \hat{f}_1(S)\hat{f}_2(S) \le C\beta^6 n^{-1-\beta^3/8}\log^2 n.$$

Proof. First notice that by Lemma 2.6 we have

$$\begin{aligned}
&\sum_{S: S\cap(\mathcal{U}_1\cup\mathcal{U}_2)\neq\emptyset} \hat{f}_1(S)\hat{f}_2(S) \\
&\le \sum_{S:(1,2)\in S} \hat{f}_1(S)\hat{f}_2(S) + \sum_{v=3}^n \sum_{S:(1,v)\in S} \hat{f}_1(S)\hat{f}_2(S) + \sum_{v=3}^n \sum_{S:(2,v)\in S} \hat{f}_1(S)\hat{f}_2(S) \\
&\le \frac{1}{n}\left(1-\frac{1}{n}\right)\mathbb{P}((1,2)\in\mathcal{P}_{f_1}\cap\mathcal{P}_{f_2}) + 2(n-2)\cdot\frac{1}{n}\left(1-\frac{1}{n}\right)\max_{u\in\{1,2\},v\neq 1,2}\mathbb{P}((u,v)\in\mathcal{P}_{f_1}\cap\mathcal{P}_{f_2}) \\
&\le \frac{1}{n}\mathbb{P}((1,2)\in\mathcal{P}_{f_1}\cap\mathcal{P}_{f_2}) + 2\mathbb{P}((1,3)\in\mathcal{P}_{f_1}\cap\mathcal{P}_{f_2}).
\end{aligned} \tag{15}$$

We first bound $\mathbb{P}((1,2) \in \mathcal{P}_{f_1} \cap \mathcal{P}_{f_2})$. Since the event that $(1,2)$ is closed is independent of the event that $(1,2)$ is pivotal for f_1 and f_2, without loss of generality we can assume that $(1,2)$ is closed. Then for $(1,2)$ to be pivotal for both f_1 and f_2, the connected components $\mathcal{C}_1$ and $\mathcal{C}_2$ of 1 and 2 respectively must satisfy

(i) $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$,

(ii) $|\mathcal{C}_1| < \beta n^{2/3} \log^{1/3} n$ and $|\mathcal{C}_2| < \beta n^{2/3} \log^{1/3} n$,

(iii) $|\mathcal{C}_1 \cup \mathcal{C}_2| \geq \beta n^{2/3} \log^{1/3} n$.

That is, setting $N = \lceil \beta n^{2/3} \log^{1/3} n \rceil$,

$$\mathbb{P}((1,2) \in \mathcal{P}_{f_1} \cap \mathcal{P}_{f_2}) \leq \mathbb{P}\Big(|\mathcal{C}_1 \cup \mathcal{C}_2| \geq N,\, |\mathcal{C}_1| < N,\, |\mathcal{C}_2| < N,\, \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset\Big).$$

By Lemma 3.4, this is at most a constant times $(N^5/n^4 + N^{3/2}/n^{5/3})e^{-N^3/(8n^2)}$.

We now move on to estimating $\mathbb{P}((1,3) \in \mathcal{P}_{f_1} \cap \mathcal{P}_{f_2})$. Just as above, we may assume that $(1,3)$ is closed; and then for $(1,3)$ to be pivotal for both f_1 and f_2, the components $\mathcal{C}_1$ and $\mathcal{C}_3$ must satisfy

(i) $\mathcal{C}_1 \cap \mathcal{C}_3 = \emptyset$,

(ii) $|\mathcal{C}_1| < \beta n^{2/3} \log^{1/3} n$ and $|\mathcal{C}_3| < \beta n^{2/3} \log^{1/3} n$,

(iii) $|\mathcal{C}_1 \cup \mathcal{C}_3| \geq \beta n^{2/3} \log^{1/3} n$,

(iv) $2 \in \mathcal{C}_1 \cup \mathcal{C}_3$.

Thus

$$\mathbb{P}((1,3) \in \mathcal{P}_{f_1} \cap \mathcal{P}_{f_2}) \leq \mathbb{P}\Big(|\mathcal{C}_1 \cup \mathcal{C}_3| \geq N,\, |\mathcal{C}_1| < N,\, |\mathcal{C}_3| < N,\, \mathcal{C}_1 \cap \mathcal{C}_3 = \emptyset,\, 2 \in \mathcal{C}_1 \cup \mathcal{C}_3\Big).$$

On the event above, clearly $|\mathcal{C}_1 \cup \mathcal{C}_3| \leq 2N$, so the probability that 2 is in $\mathcal{C}_1 \cup \mathcal{C}_3$ is at most $2N/n$. Therefore

$$\mathbb{P}((1,3) \in \mathcal{P}_{f_1} \cap \mathcal{P}_{f_2}) \leq \frac{2N}{n}\mathbb{P}\Big(|\mathcal{C}_1 \cup \mathcal{C}_3| \geq N,\, |\mathcal{C}_1| < N,\, |\mathcal{C}_3| < N,\, \mathcal{C}_1 \cap \mathcal{C}_3 = \emptyset\Big).$$

Applying Lemma 3.4 again, we get

$$\mathbb{P}((1,3) \in \mathcal{P}_{f_1} \cap \mathcal{P}_{f_2}) \leq c\frac{N}{n}(N^5/n^4 + N^{3/2}/n^{5/3})e^{-N^3/(8n^2)}$$

for some finite constant c.

Plugging these estimates back into (15), we have

$$\sum_{S: S \cap (\mathcal{U}_1 \cup \mathcal{U}_2) \neq \emptyset} \hat{f}_1(S)\hat{f}_2(S) \leq c\Big(\frac{1}{n} + \frac{N}{n}\Big)(N^5/n^4 + N^{3/2}/n^{5/3})e^{-N^3/(8n^2)}.$$

Recalling that $N = \lceil \beta n^{2/3} \log^{1/3} n \rceil$ and simplifying, we get

$$\sum_{S:S\cap(\mathcal{U}_1\cup\mathcal{U}_2)\neq\emptyset} \hat{f}_1(S)\hat{f}_2(S) \leq c'\Big(\frac{\beta^6 \log^2 n}{n} + \frac{\beta^{5/2}\log^{5/6} n}{n}\Big)n^{-\beta^3/8},$$

and the result follows. □

Now we deal with the Fourier coefficients $\hat{f}_1(S)\hat{f}_2(S)$ where $S \cap (\mathcal{U}_1 \cup \mathcal{U}_2) = \emptyset$. Notice that by symmetry we have that if $S \cap (\mathcal{U}_1 \cup \mathcal{U}_2) = \emptyset$, then $\hat{f}_1(S) = \hat{f}_2(S)$ and so

$$\sum_{|S|>0;S\cap(\mathcal{U}_1\cup\mathcal{U}_2)=\emptyset} \frac{e^{-\delta|S|}}{|S|}\hat{f}_1(S)\hat{f}_2(S) \leq \sum_{|S|>0;S\cap\mathcal{U}_1=\emptyset} \frac{e^{-\delta|S|}}{|S|}\hat{f}_1(S)^2. \tag{16}$$

To estimate the sum on the right hand side we use a revealment algorithm, implementing Theorem 2.2. Any sensible algorithm will do; we can reveal all of the edges emanating from vertex 1 without concern, and thereafter the lack of geometry in the graph simplifies the problem.

The algorithm A that we choose to use is the breadth first search and is described as follows. At each step $i \geq 0$ we have an ordered list of vertices $\mathcal{S}_i$, which is the list of vertices that the algorithm already knows are in $\mathcal{C}_1$. We begin from $\mathcal{S}_0 = \{1\}$. At each step i, if $|\mathcal{S}_i| \geq \beta n^{2/3}\log^{1/3} n$ then we terminate and declare that $f_1(\omega) = 1$; or if $|\mathcal{S}_i| < i$ then we terminate the algorithm and declare that $f_1(\omega) = 0$. Otherwise we take the ith element v_i of $\mathcal{S}_i$, and reveal $\omega((v_i, w))$ for all $w \notin \mathcal{S}_i$. If $\omega((v_i, w)) = 1$ then we add w to the end of the list, and once we have revealed all such edges (in some arbitrary order), the resulting list is then $\mathcal{S}_{i+1}$.

Clearly the algorithm must terminate by step $\lceil \beta n^{2/3}\log^{1/3} n \rceil$. Recall that

$$\mathcal{R}_{\mathcal{U}_1} = \max_{e\notin\mathcal{U}_1} \mathbb{P}(A \text{ reveals } \omega(e)).$$

Lemma 4.4. *Let A be the breadth first search described above. There exists a finite constant C such that for all large n,*

$$\mathcal{R}_{\mathcal{U}_1} \leq C\beta^{7/2} n^{-2/3}\log^{7/6} n.$$

Proof. Let τ be the step at which the algorithm A terminates. For any edge $e = (v, w) \notin \mathcal{U}_1$, the probability that we reveal $\omega(e)$ is at most the probability that either v or w appears in $\mathcal{S}_{\tau-1}$. For any $v, u \neq 1$ we have $\mathbb{P}(v \in \mathcal{S}_{\tau-1}) = \mathbb{P}(u \in \mathcal{S}_{\tau-1})$, and thus

$$\begin{aligned}\mathbb{P}(A \text{ reveals } \omega(e)) &\leq 2\mathbb{P}(v \in \mathcal{S}_{\tau-1}) \\ &= \frac{2}{n-1}\mathbb{E}\Big[\sum_{u\neq 1} \mathbb{1}_{\{u\in\mathcal{S}_{\tau-1}\}}\Big] = \frac{2(\mathbb{E}[|\mathcal{S}_{\tau-1}|]-1)}{n-1} \leq \frac{2}{n}\mathbb{E}[|\mathcal{S}_{\tau-1}|].\end{aligned} \tag{17}$$

Let $N = \lceil \beta n^{2/3}\log^{1/3} n \rceil$. It is easy to see from the description of the algorithm that we always have $\mathcal{S}_{\tau-1} \subset \mathcal{C}_1$, and if $|\mathcal{C}_1| \geq N$ then $|\mathcal{S}_{\tau-1}| \leq N$. Combining this observation with (17), then applying Proposition 3.1 and Lemma 3.2, we get that

$$\begin{aligned}\mathbb{P}(A \text{ reveals } \omega(e)) &\leq \sum_{k=1}^{N} \frac{2k}{n}\mathbb{P}(|\mathcal{C}_1| = k) + \frac{2N}{n}\mathbb{P}(|\mathcal{C}_1| \geq N) \\ &\leq c \sum_{k=1}^{\lfloor n^{2/3} \rfloor} \frac{k}{n}k^{-3/2} + c \sum_{k=\lfloor n^{2/3}\rfloor+1}^{N} \frac{k}{n}\frac{k^{3/2}}{n^2} + c\frac{N}{n}\frac{n^{-\beta^3/8}}{n^{1/3}\log^{1/6} n} \\ &\leq c' n^{-2/3} + c'\frac{N^{7/2}}{n^3} + c' n^{-\beta^3/8-2/3}\log^{1/6} n\end{aligned}$$

for some finite constants c, c'. For large n this is at most a constant times $\beta^{7/2}n^{-2/3}\log^{7/6} n$. $\square$

Now we apply Theorem 2.2 and Lemma 4.4 to estimate the Fourier coefficients $\hat{f}_1(S)\hat{f}_2(S)$ when $S \cap (\mathcal{U}_1 \cup \mathcal{U}_2) = \emptyset$.

Lemma 4.5. *There exists a finite constant C such that for any $\delta > 0$ and all large n,*

$$\sum_{\substack{|S|>0;\\ S\cap(\mathcal{U}_1\cup\mathcal{U}_2)=\emptyset}} \frac{e^{-\delta|S|}}{|S|}\hat{f}_1(S)\hat{f}_2(S) \leq C\delta^{-1}\mathbb{E}[f_1]\beta^{7/2}n^{-2/3}\log^{7/6} n.$$

Proof. First recall that by (16) we have

$$\sum_{\substack{|S|>0;\\ S\cap(\mathcal{U}_1\cup\mathcal{U}_2)=\emptyset}} \frac{e^{-\delta|S|}}{|S|}\hat{f}_1(S)\hat{f}_2(S) \leq \sum_{\substack{|S|>0;\\ S\cap\mathcal{U}_1=\emptyset}} \frac{e^{-\delta|S|}}{|S|}\hat{f}_1(S)^2 = \sum_{k=1}^{\binom{n}{2}} \frac{1}{k}e^{-\delta k} \sum_{\substack{|S|=k;\\ S\cap\mathcal{U}_1=\emptyset}} \hat{f}_1(S)^2. \tag{18}$$

By Theorem 2.2, this is at most

$$\sum_{k=1}^{\binom{n}{2}} e^{-\delta k}\mathcal{R}_{\mathcal{U}_1}\mathbb{E}[f_1^2].$$

Since f_1 takes values in $\{0,1\}$, $\mathbb{E}[f_1^2] = \mathbb{E}[f_1]$, and by Lemma 4.4, $\mathcal{R}_{\mathcal{U}_1} \leq C\beta^{7/2}n^{-2/3}\log^{7/6} n$. Finally, note that

$$\sum_{k=1}^{\infty} e^{-\delta k} = \frac{e^{-\delta}}{1-e^{-\delta}} \leq \delta^{-1}.$$

Combining these three observations gives the desired result. $\square$

4.3 Completing the proof of Theorem 1.1 for small β

Recall that we began by defining $Z_v = \int_0^1 \mathbb{1}_{\{|\mathcal{C}_v(t)|\in I\}}dt$ where $I = [\beta n^{2/3}\log^{1/3} n, 2\beta n^{2/3}\log^{1/3} n] \cap \mathbb{N}$. From (6) we know that

$$\mathbb{P}\Big(\sup_{t\in[0,1]} |\mathcal{C}(t)| > \beta n^{2/3}\log^{1/3} n\Big) \geq \frac{n^2\mathbb{E}[Z_1]^2}{n\mathbb{E}[Z_1^2] + n(n-1)\mathbb{E}[Z_1Z_2]}.$$

Lemma 4.1 told us that if $\beta^3 < 16/3$ then $\frac{\mathbb{E}[Z_1^2]}{n\mathbb{E}[Z_1]^2} \to 0$ as $n \to \infty$, in which case we get that

$$\liminf_{n\to\infty} \mathbb{P}\Big(\sup_{t\in[0,1]} |L_n(t)| > n^{2/3}\log^\beta n\Big) \geq \liminf_{n\to\infty} \frac{\mathbb{E}[Z_1]^2}{\mathbb{E}[Z_1 Z_2]}. \tag{19}$$

We saw in (8) that

$$\mathbb{E}[Z_1 Z_2] = \int_0^1 \int_0^1 \mathbb{P}(|\mathcal{C}_1(s))| \in I; |\mathcal{C}_2(t)| \in I)\,\mathrm{d}t\,\mathrm{d}s.$$

Let $P = \mathbb{P}(|\mathcal{C}_1| \geq \beta n^{2/3}\log^{1/3} n)$. Lemma 4.2 gives

$$\int_0^1 \int_0^1 \mathbb{P}(|\mathcal{C}_1(s)| \in I; |\mathcal{C}_2(t)| \in I)\mathbb{1}_{\{|t-s|\leq\delta\}}\,\mathrm{d}t\,\mathrm{d}s \leq 2\delta P^2 + \frac{4\beta\log^{1/3} n}{n^{1/3}}\delta P + 2\delta^2 P.$$

By Proposition 3.1, we have $P \leq n^{-1/3-\beta^3/8}$ for large n, so we get (for large n)

$$\begin{aligned}\int_0^1 \int_0^1 &\mathbb{P}(|\mathcal{C}_1(s)| \in I; |\mathcal{C}_2(t)| \in I)\mathbb{1}_{\{|t-s|\leq n^{-1/3+\varepsilon}\}}\,\mathrm{d}t\,\mathrm{d}s \\ &\leq 2\delta n^{-2/3-\beta^3/4} + 4\beta\delta(\log^{1/3} n)n^{-2/3-\beta^3/8} + 2\delta^2 n^{-1/3-\beta^3/8} \\ &\leq 5\beta\delta(\log^{1/3} n)n^{-2/3-\beta^3/8} + 2\delta^2 n^{-1/3-\beta^3/8}\end{aligned} \tag{20}$$

To estimate the integral when $|t-s| > n^{-1/3}$, we begin with (14), which says that

$$\int_0^1 \int_0^1 \mathbb{P}(|\mathcal{C}_1(s)| \in I; |\mathcal{C}_2(t)| \in I)\mathbb{1}_{\{|t-s|>\delta\}}\,\mathrm{d}t\,\mathrm{d}s \leq \mathbb{E}[Z_1]^2 + 2\sum_{|S|>0} \frac{e^{-\delta|S|}}{|S|}\hat{f}_1(S)\hat{f}_2(S).$$

We now apply Lemmas 4.3 and 4.5, which tell us respectively that for large n,

$$\sum_{S:S\cap(\mathcal{U}_1\cup\mathcal{U}_2)\neq\emptyset} \hat{f}_1(S)\hat{f}_2(S) \leq C\beta^6 n^{-1-\beta^3/8}\log^2 n$$

and

$$\sum_{\substack{S:|S|>0;\\ S\cap(\mathcal{U}_1\cup\mathcal{U}_2)=\emptyset}} \frac{e^{-\delta|S|}}{|S|}\hat{f}_1(S)\hat{f}_2(S) \leq C\delta^{-1}\mathbb{E}[f_1]\beta^{7/2}n^{-2/3}\log^{7/6} n.$$

for some finite constant C. Combining these three equations and noting that (by Proposition 3.1) $\mathbb{E}[f_1] \leq n^{-1/3-\beta^3/8}$, we get

$$\begin{aligned}\int_0^1 \int_0^1 &\mathbb{P}(|\mathcal{C}_1(s)| \in I; |\mathcal{C}_2(t)| \in I)\mathbb{1}_{\{|t-s|>n^{-1/3+\varepsilon}\}}\mathrm{d}t\mathrm{d}s \\ &\leq \mathbb{E}[Z_1]^2 + 2C\beta^6 n^{-1-\beta^3/8}\log^2 n + 2C\delta^{-1}\beta^{7/2}n^{-1-\beta^3/8}\log^{7/6} n.\end{aligned} \tag{21}$$

Combining (20) with (21) and plugging back into (8), we get

$$\mathbb{E}[Z_1Z_2] \le \mathbb{E}[Z_1]^2 + 5\beta\delta(\log^{1/3} n)n^{-2/3-\beta^3/8} + 2\delta^2 n^{-1/3-\beta^3/8} \\ + 2C\beta^6 n^{-1-\beta^3/8}\log^2 n + 2C\delta^{-1}\beta^{7/2}n^{-1-\beta^3/8}\log^{7/6} n.$$

Choosing $\delta = n^{-2/9}$, for large n we have

$$\mathbb{E}[Z_1Z_2] \le \mathbb{E}[Z_1]^2 + 2C\beta^6 n^{-7/9-\beta^3/8}\log^2 n.$$

By Proposition 3.1 we know that $\mathbb{E}[Z_1] \ge cn^{-\beta^3/8-1/3}\log^{-1/6} n$ for some constant $c > 0$, so we get

$$\frac{\mathbb{E}[Z_1Z_2]}{\mathbb{E}[Z_1]^2} \le 1 + c'\beta^6 n^{-1/9+\beta^3/8}\log^2 n$$

for some finite constant c'. For $\beta < 2/3^{2/3}$, the above quantity tends to 1 as $n \to \infty$, giving

$$\liminf_{n\to\infty} \frac{\mathbb{E}[Z_1]^2}{\mathbb{E}[Z_1Z_2]} \ge 1.$$

Therefore by (19), for any $\beta < 2/3^{2/3}$,

$$\liminf_{n\to\infty} \mathbb{P}\Big(\sup_{t\in[0,1]} |L_n(t)| > \beta n^{2/3}\log^{1/3} n\Big) = 1.$$

We have shown that exceptional times exist with high probability for any $\beta < 2/3^{2/3}$, and to complete the proof of Theorem 1.1 it remains to show that with high probability there are no such times for any $\beta \ge 2/3^{1/3}$.

5 No exceptional times when $\beta \ge 2/3^{1/3}$

Fix $\beta > 0$. For $i \in \{0, \dots, \lfloor n^{1/3}\rfloor\}$, consider the event

$$\mathcal{E}_i := \{\exists t \in [in^{-1/3}, (i+1)n^{1/3}) : |L_n(t)| > \beta n^{2/3}\log^{1/3} n\}.$$

The probability that an edge e is turned on at any time in $[in^{-1/3}, (i+1)n^{1/3})$ is at most $1/n + (1-e^{-n^{-1/3}})/n \le (1+n^{-1/3})/n$. Therefore for each i,

$$\mathbb{P}(\mathcal{E}_i) \le \mathbf{P}_{n,n^{-1}+n^{-4/3}}(|L_n| > \beta n^{2/3}\log^{1/3} n)$$

where we recall that $\mathbf{P}_{n,p}$ is the law of an $\mathrm{ER}(n,p)$. Applying Proposition 3.1 with $\lambda = 1$, we get that for large n,

$$\mathbb{P}(\mathcal{E}_i) \le \beta^{-3/2} n^{-\beta^3/8}\log^{-1/2} n,$$

so by a union bound,

$$\mathbb{P}(\exists t \in [0,1] : |L_n(t)| > \beta n^{2/3}\log^{1/3} n) \le \beta^{-3/2} n^{1/3-\beta^3/8}\log^{-1/2} n.$$

This tends to zero as $n \to \infty$ if $\beta \ge 2/3^{1/3}$, which shows that with high probability there are no exceptional times in this regime. This completes the proof of Theorem 1.1.

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