Deep factorisation of the stable process II; potentials and applications

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Abstract

Here we propose a different perspective of the *deep factorisation* in [20] based on determining potentials. Indeed, we factorise the inverse of the MAP–exponent associated to a stable process via the Lamperti–Kiu transform. Here our factorisation is completely independent from the derivation in [20], moreover there is no clear way to invert the factors in [20] to derive our results. Our method gives direct access to the potential densities of the ascending and descending ladder MAP of the Lamperti-stable MAP in closed form.

In the spirit of the interplay between the classical Wiener–Hopf factorisation and fluctuation theory of the underlying Lévy process, our analysis will produce a collection of of new results for stable processes. We give an identity for the point of closest reach to the origin for a stable process with index $\alpha \in (0,1)$ as well as and identity for the point of furthest reach before absorption at the origin for a stable process with index $\alpha \in (1,2)$. Moreover, we show how the deep factorisation allows us to compute explicitly the stationary distribution of stable processes multiplicatively reflected in such a way that it remains in the strip $[-1,1]$.

1 Introduction and main results

Let $(X, \mathbb{P}_x)$ be a one-dimensional Lévy process started at $x \in \mathbb{R}$. Suppose that, when it exists, we write ψ for its Laplace exponent, that is to say, $\psi(z) := t^{-1} \log \mathbb{E}_0[e^{zX_t}]$ for all $z \in \mathbb{C}$ such that the right-hand side exists. An interesting aspect of the characteristic exponent of Lévy processes is that they can be written as a product of the so called Weiner–Hopf factors, see for example [21, Theorem 6.15]. This means that there exists two Bernstein functions κ and $\hat{\kappa}$ (see [26] for a definition) such that

$$-\psi(i\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta), \qquad \theta \in \mathbb{R}. \tag{1}$$

There are, of course, many functions κ and $\hat{\kappa}$ which satisfy (1). However imposing the requirements that the functions κ and $\hat{\kappa}$ must be Bernstein functions which analytically extend in θ to the upper and lower half plane of $\mathbb{C}$, respectively, means that the factorisation in (1) is unique up to a multiplicative constant.

The problem of finding such factors has considerable interest since, probablistically speaking, the Bernstein functions κ and $\hat{\kappa}$ are the Laplace exponents of the ascending and descending ladder height processes respectively. The ascending and descending ladder height processes, say $(h_t : t \geq 0)$ and $(\hat{h}_t : t \geq 0)$, are subordinators that correspond to a time change of $\overline{X}_t := \sup_{s \leq t} X_s$, $t \geq 0$ and $-\underline{X}_t := -\inf_{s \leq t} X_s$, $t \geq 0$, and therefore have the same range, respectively. Additional information comes from the exponents in that they also provide information about the potential measures

associated to their respective ladder height processes. So for example, $U(dx) := \int_0^\infty \mathbb{P}(h_t \in dx)dt$, $x \geq 0$, has Laplace transform given by $1/\kappa$. A similar identity hold for $\widehat{U}$, the potential of $\hat{h}$.

These potential measures appear in a wide variety of fluctuation identities. Perhaps the most classical example concerns the stationary distribution of the process reflected in its maximum, $\overline{X}_t - X_t$, $t \geq 0$ in the case that $\lim_{t\to\infty} X_t = \infty$. In that case, we may take $\lim_{t\to\infty} \mathbb{P}_x(\overline{X}_t - X_t \in dx) = \kappa(0)\widehat{U}(dx)$; c.f. [27]. The ladder height potential measures also feature heavily in first passage identities that describe the joint law of the triple $(X_{\tau_x^+}, X_{\tau_x^+-}, \overline{X}_{\tau_x^+-})$, where $\tau_x^+ = \inf\{t > 0 : X_t > x\}$ and $x > 0$; cf. [13]. Specifically, one has, for $u > 0$, $v \geq y$ and $y \in [0, x]$,

$$\mathbb{P}(X_{\tau_x^+} - x \in du, x - X_{\tau_x^+-} \in dv, x - \overline{X}_{\tau_x^+-} \in dy) = U(x - dy)\widehat{U}(dv - y)\Pi(du + v),$$

where Π is the Lévy measure of X. A third example we give here pertains to the much more subtle property of increase times. Specifically, an increase time is a (random) time $t > 0$ at which there exists a (random) $\varepsilon > 0$ such that $X_{t'} \leq X_t \leq X_{t''}$ for all $t' \in [t - \varepsilon, t]$ and $t'' \in [t, t + \varepsilon]$. The existence of increase times occurs with probability 0 or 1. It is known that under mild conditions, increase times exist almost surely if and only if $\int_{0+} \widehat{U}(x)^{-1}U(dx) < 0$. See, for example, [14] and references therein.

Within the class of Lévy processes which experience jumps of both sign, historically there have been very few explicit cases of the Wiener–Hopf factorisation identified within the literature. More recently, however, many new cases have emerged hand-in-hand with new characterisations and distributional identities of path functionals of stable processes; see e.g. the summary in [21, Section 6.5 and Chapter 13]. A Lévy process $(X, \mathbb{P}_x)$ is called an α-stable process for $\alpha \in (1, 2]$ if for every $c > 0$, $(cX_{c^{-\alpha}t} : t \geq 0)$ under $\mathbb{P}_x$ has the same law as $(X, \mathbb{P}_{cx})$. The case when $\alpha = 2$ corresponds to Brownian motion, which we henceforth exclude from all subsequent discussion. It is known that the Laplace exponent can be parametrised so that

$$\psi(i\theta) = -|\theta|^\alpha(e^{i\pi\alpha(1/2-\rho)}\mathbb{1}_{\{\theta \geq 0\}} + e^{-i\pi\alpha(1/2-\hat{\rho})}\mathbb{1}_{\{\theta < 0\}}), \qquad \theta \in \mathbb{R},$$

where $\rho := \mathbb{P}(X_1 \geq 0)$ is the positivity parameter and $\hat{\rho} = 1 - \rho$. In that case, the two factors are easily identified as $\kappa(\lambda) = \lambda^{\alpha\rho}$ and $\hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}$ for $\lambda \geq 0$, with associated potentials possessing densities proportional to $x^{\alpha\rho-1}$ and $x^{\alpha\hat{\rho}-1}$ respectively for $x \geq 0$. In part, this goes a long way to explaining why so many fluctuation identities are explicitly available for stable processes.

In this paper, our objective is to produce a new explicit characterisation of a second type of explicit Weiner–Hopf factorisation embedded in the α-stable process, the so-called 'deep factorisation' first considered in [20], through its representation as a real-valued self-similar Markov process. In the spirit of the interplay between the classical Wiener–Hopf factorisation and fluctuation theory of the underlying Lévy process, our analysis will produce a collection of new results for stable processes which are based on identities for potentials derived from the deep factorisation. Before going into detail regarding our results concerning the deep factorisation, let us first present some of the new results we shall obtain *en route* for stable processes.

1.1 Results on fluctuations of stable processes

The first of our results on stable processes concerns the 'point of closest reach' to the origin for stable processes with index $\alpha \in (0, 1)$. Recall that for this index range, the stable process does not hit points and, moreover, $\liminf_{t\to\infty} |X_s| = \infty$. Hence, either on the positive or negative side of the origin, the path of the stable process has a minimal radial distance. Moreover, this distance is

achieved at the unique time $\underline{m}$ such that $|X_t| \geq |X_{\underline{m}}|$ for all $t \geq 0$. Note, uniqueness follows thanks to regularity of X for both $(0, \infty)$ and $(-\infty, 0)$.

Proposition 1.1 (Point of closest reach)**.** *Suppose that $\alpha \in (0, 1)$, then for $x > 0$ and $|z| \leq x$,*

$$\mathbb{P}_x(X_{\underline{m}} \in dz) = \frac{\Gamma(1 - \alpha\rho)}{\Gamma(1 - \alpha)\Gamma(\alpha\hat{\rho})} \frac{x + z}{|2z|^\alpha} (x - |z|)^{\alpha\hat{\rho}-1} (x + |z|)^{\alpha\rho-1} dz.$$

In the case that $\alpha = 1$, the stable process does not hit points and we have that $\limsup_{t\to\infty} |X_t| = \infty$ and $\liminf_{t\to\infty} |X_t| = 0$ and hence it is difficult to produce a result in the spirit of the previous theorem. However, when $\alpha \in (1, 2)$, the stable process will hit all points almost surely, in particular $\tau^{\{0\}} := \inf\{t > 0 : X_t = 0\}$ is $\mathbb{P}_x$-almost surely finite for all $x \in \mathbb{R}$. This allows us to talk about the 'point of furthest' reach until absorption at the origin. To this end, we define $\overline{m}$ to be the unique time such that $|X_t| \leq |X_{\overline{m}}|$ for all $t \leq \tau^{\{0\}}$. Note again, that uniqueness is again guaranteed by regularity of the upper and lower half line for the stable process.

Proposition 1.2 (Point of furthest reach)**.** *Suppose that $\alpha \in (1, 2)$, then for each $x > 0$ and $|z| > x$,*

$$\mathbb{P}_x(X_{\overline{m}} \in dz) = \frac{\alpha - 1}{2|z|^\alpha} \left(|x + z|(|z| - x)^{\alpha\rho-1}(|z| + x)^{\alpha\hat{\rho}-1} \right.$$
$$\left. -(\alpha - 1)x \int_1^{|z|/x} (t - 1)^{\alpha\rho-1}(t + 1)^{\alpha\hat{\rho}-1} dt \right).$$

Finally we are also interested in *radially reflected stable processes*:

$$R_t = \frac{X_t}{M_t \vee 1} \qquad t \geq 0,$$

where $M_t = \sup_{s\leq t} |X_s|$, $t \geq 0$. It is easy to verify, using the scaling and Markov properties, that for $t, s \geq 0$,

$$M_{t+s} = |X_t| \left(\frac{M_t}{|X_t|} \vee \widetilde{M}_{s|X_t|^{-\alpha}} \right), \qquad X_{t+s} = |X_t|\widetilde{X}_{s|X_t|^{-\alpha}},$$

where $(\widetilde{M}, \widetilde{X})$ is such that, for all bounded measurable functions f,

$$\mathbb{E}[f(\widetilde{M}_s, \widetilde{X}_s)|\sigma(X_u : u \leq t)] = g(\text{sign}(X_t)),$$

where $g(y) = \mathbb{E}_y[f(M_s, X_s)]$. It follows that, whilst the process R is not Markovian, the pair (R, M) is a strong Markov process. In forthcoming work, in the spirit of Chaumont et al. [9], we shall demonstrate how an excursion theory can be developed for the pair (R, M). In particular, one may build a local time process which is supported on the closure of the times $\{t : |R_t| = 1\}$. The times that are not in this supporting set form a countable union of disjoint intervals during which R executes an excursion *into* the interval $(-1, 1)$ (i.e. the excursion begins on the boundary $(-1, 1)$ and runs until existing this interval). We go no further into the details of this excursion theory here. However, it is worthy of note that one should expect an ergodic limit of the process R which is bounded in $[-1, 1]$. The following result demonstrates this prediction in explicit detail.
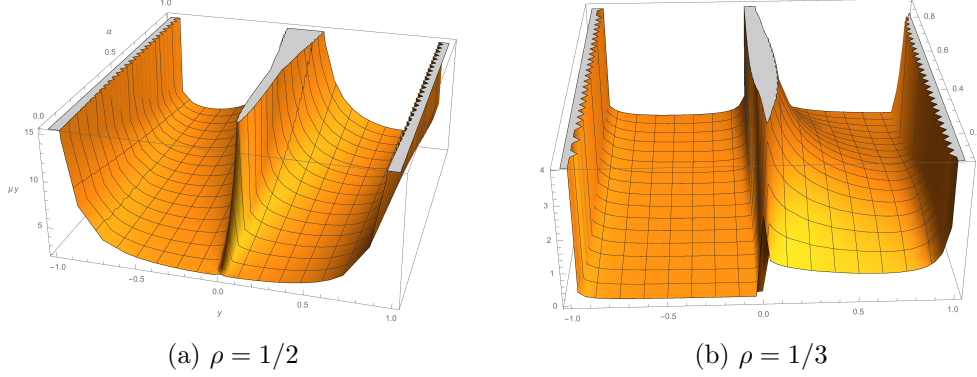
(a) $\rho = 1/2$ (b) $\rho = 1/3$

Figure 1: The stationary distribution of the reflected process for two different values of ρ. Note that the concentration of mass at the values $1, -1$ is a consequence of the time that X spends in close proximity to M. Moreover, the concentration of mass at 0 is a result of the division of X by large values of M during periods that it crosses from a positive radial extrema to a negative radial extremum and vice versa.

Theorem 1.3 (Stationary distribution of the radially reflected process). *Suppose that $\alpha \in (0,1)$. Let $x \in (-1,1)$, then under $\mathbb{P}_x$, R has a limiting distribution μ, concentrated on $[-1,1]$, given by*

$$\frac{\mu(dy)}{dy}$$
$$= \begin{cases} c_{\alpha,\rho}\sin(\alpha\pi\rho)y^{-\alpha}(1-y)^{\alpha\hat{\rho}-1}(1+y)^{\alpha\rho} + c_{\alpha,\rho}\sin(\alpha\pi\hat{\rho})y^{-\alpha}(1-y)^{\alpha\rho}(1+y)^{\alpha\hat{\rho}-1} & \text{if } y \in [0,1] \\[2mm] c_{\alpha,\hat{\rho}}\sin(\alpha\pi\hat{\rho})|y|^{-\alpha}(1-|y|)^{\alpha\rho-1}(1+|y|)^{\alpha\hat{\rho}} + c_{\alpha,\hat{\rho}}\sin(\alpha\pi\rho)|y|^{-\alpha}(1-|y|)^{\alpha\hat{\rho}}(1+|y|)^{\alpha\rho-1} & \text{if } y \in [-1,0). \end{cases}$$

where

$$c_{\alpha,\rho} = \frac{\Gamma(1-\alpha\rho)}{2^{\alpha}[\sin(\alpha\pi\rho) + \sin(\alpha\pi\hat{\rho})]\Gamma(1-\alpha)\Gamma(\alpha\hat{\rho})},$$

with a similar definition for $c_{\alpha,\hat{\rho}}$. (See Figure 1 which has two examples of this density for different values of ρ.)

1.2 Results on the deep factorisation of stable processes

In order to present our results on the deep factorisation, we must first look the Lamperti–Kiu representation of real self-similar Markov processes, and in particular for the case of a stable process.

A Markov process $(X, \mathbb{P}_x)$ is called a real self-similar Markov processes (rssMp) of index $\alpha > 0$ if, for every $c > 0$, $(cX_{c^{-\alpha}t} : t \geq 0)$ has the same law as X. In particular, an α-stable Lévy process is an example of an rssMp, in which case $\alpha \in (0,2)$. Recall, every rssMp can be written in terms of what is referred to as a Markov additive process (MAP) (ξ, J). The details of this can be found in Section 2. Essentially (ξ, J) is a certain (possibly killed) Markov process taking values in $\mathbb{R} \times \{1, -1\}$ and is completely characterised by its so-called MAP–exponent $\boldsymbol{F}$ which, when it exists, is a function

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mapping $\mathbb{C}$ to 2×2 complex valued matrices[1], which satisfies,

$$(e^{\boldsymbol{F}(z)t})_{i,j} = \mathbf{E}_{0,i}[e^{z\xi(t)}; J_t = j].$$

The Lamperti–Kiu transformation states that for each rssMp X, there exists a MAP (ξ, J) such that

$$X_t \mathbb{1}_{\{t<\tau^{\{0\}}\}} = |x|e^{\xi_{\varphi(|x|^{-\alpha}t)}} J_{\varphi(|x|^{-\alpha}t)} \tag{2}$$

where $\tau^{\{0\}} := \inf\{t \geq 0 : X_t = 0\}$ and

$$\varphi(t) := \inf\left\{s > 0 : \int_0^s e^{\alpha\xi_u}\,du > t\right\}.$$

We use the convention that $\inf \emptyset = \infty$ and $e^{\xi_\infty} = 0$. We will denote the law of (ξ, J) started from (x, i) by $\mathbf{P}_{x,i}$. Conversely, every Markov additive process (ξ, J) on $\mathbb{R} \times \{1, -1\}$ defines a real–valued self–similar Markov process via (2).

Thus, in the case $X_t \geq 0$ for all $t \geq 0$, we have that $J_t = 1$ for all $t \geq 0$. In this special case the transformation is known as the Lamperti transform and the process ξ is a Lévy process. The Lamperti transformation, introduced in [24], has been studied intensively, see for example [21, Chapter 13] and references therein. In particular, the law and the Weiner–Hopf factorisation of ξ is known in many cases, for example [21, Section 13.4] and [7].

Conversely, very little is known about the general case. In this paper, we shall consider the case when X is a α–stable process (unit first hitting the origin in the case that $\alpha \in (1, 2)$) and note that in that case (ξ, J) is not necessarily a Lévy process. In this case the MAP–exponent is known to be

$$\boldsymbol{F}(z) = \begin{pmatrix} -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho}-z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho-z)\Gamma(1-\alpha\rho+z)} \end{pmatrix}, \tag{3}$$

for $\mathrm{Re}(z) \in (-1, \alpha)$, and the associated process is called the *Lamperti-stable* MAP.

Similar to the case of Lévy processes, we can define $\boldsymbol{\kappa}$ and $\hat{\boldsymbol{\kappa}}$ as the MAP–exponent of the ascending and descending ladder height process for (ξ, J), see Section 2 for more details. In the case when X is an α-stable process, we have that, for each $\lambda \in \mathbb{R}$,

$$-\boldsymbol{F}(i\lambda) = \boldsymbol{\Delta}_\pi^{-1}\hat{\boldsymbol{\kappa}}(i\lambda)^T \boldsymbol{\Delta}_\pi \boldsymbol{\kappa}(-i\lambda), \tag{4}$$

where

$$\boldsymbol{\Delta}_\pi := \begin{pmatrix} \sin(\alpha\pi\rho) & 0 \\ 0 & \sin(\alpha\pi\hat{\rho}). \end{pmatrix}. \tag{5}$$

The factorisation in (4) can be found in [15] and [16] for example. Our main result identifies the inverse factors $\boldsymbol{\kappa}^{-1}$ and $\hat{\boldsymbol{\kappa}}^{-1}$ explicitly.

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[1]Here and throughout the paper the matrix entries are arranged by

$$A = \begin{pmatrix} A_{1,1} & A_{1,-1} \\ A_{-1,1} & A_{-1,-1} \end{pmatrix}.$$

Theorem 1.4. *Suppose that X is an α-stable process then we have that the factors $\boldsymbol{\kappa}^{-1}$ and $\hat{\boldsymbol{\kappa}}^{-1}$ are given as follows. For $a, b, c \in \mathbb{R}$, define*

$$\Psi(a, b, c) := \int_0^1 u^a (1-u)^b (1+u)^c du. \tag{6}$$

Case $\alpha \in (0, 1)$: For all $\lambda \geq 0$,

$$\boldsymbol{\kappa}^{-1}(\lambda)$$

$$= \frac{1}{2^\alpha \Gamma(1-\alpha)} \begin{pmatrix} \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)} \Psi(\lambda-1, \alpha\rho-1, \alpha\hat\rho) & \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)} \Psi(\lambda-1, \alpha\rho, \alpha\hat\rho-1) \\ \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)} \Psi(\lambda-1, \alpha\hat\rho, \alpha\rho-1) & \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)} \Psi(\lambda-1, \alpha\hat\rho-1, \alpha\rho) \end{pmatrix}$$

and

$$\hat{\boldsymbol{\kappa}}^{-1}(\lambda)$$

$$= \frac{1}{2^\alpha \Gamma(1-\alpha)} \begin{pmatrix} \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)} \Psi(\lambda-\alpha, \alpha\hat\rho-1, \alpha\rho) & \frac{\sin(\alpha\pi\hat\rho)\Gamma(1-\alpha\rho)}{\sin(\alpha\pi\rho)\Gamma(\alpha\hat\rho)} \Psi(\lambda-\alpha, \alpha\hat\rho, \alpha\rho-1) \\ \frac{\sin(\alpha\pi\rho)\Gamma(1-\alpha\hat\rho)}{\sin(\alpha\pi\hat\rho)\Gamma(\alpha\rho)} \Psi(\lambda-\alpha, \alpha\rho, \alpha\hat\rho-1) & \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)} \Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat\rho) \end{pmatrix}.$$

Case $\alpha = 1$: For all $\lambda \geq 0$

$$\boldsymbol{\kappa}^{-1}(\lambda) = \hat{\boldsymbol{\kappa}}^{-1}(\lambda) = \begin{pmatrix} \Psi(\lambda-1, -1/2, 1/2) & \Psi(\lambda-1, 1/2, -1/2) \\ \Psi(\lambda-1, 1/2, -1/2) & \Psi(\lambda-1, -1/2, 1/2) \end{pmatrix}.$$

Case $\alpha \in (1, 2]$: For all $\lambda \geq 0$

$$\boldsymbol{\kappa}^{-1}(\lambda) = \frac{\alpha-1}{2} \begin{pmatrix} \Psi(\lambda-1, \alpha\rho-1, \alpha\hat\rho) & \Psi(\lambda-1, \alpha\rho, \alpha\hat\rho-1) \\ \Psi(\lambda-1, \alpha\hat\rho, \alpha\rho-1) & \Psi(\lambda-1, \alpha\hat\rho-1, \alpha\rho) \end{pmatrix}$$

$$- \frac{(\alpha-1)^2}{2(\lambda+\alpha-1)} \begin{pmatrix} \Psi(\lambda-1, \alpha\rho-1, \alpha\hat\rho-1) & \Psi(\lambda-1, \alpha\rho-1, \alpha\hat\rho-1) \\ \Psi(\lambda-1, \alpha\hat\rho-1, \alpha\rho-1) & \Psi(\lambda-1, \alpha\hat\rho-1, \alpha\rho-1) \end{pmatrix}$$

and

$$\hat{\boldsymbol{\kappa}}^{-1}(\lambda) = \frac{\alpha-1}{2} \begin{pmatrix} \Psi(\lambda-\alpha, \alpha\hat\rho-1, \alpha\rho) & \frac{\sin(\alpha\pi\hat\rho)}{\sin(\alpha\pi\rho)} \Psi(\lambda-\alpha, \alpha\hat\rho, \alpha\rho-1) \\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat\rho)} \Psi(\lambda-\alpha, \alpha\rho, \alpha\hat\rho-1) & \Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat\rho) \end{pmatrix}$$

$$- \frac{(\alpha-1)^2}{2(\lambda+\alpha-1)} \begin{pmatrix} \Psi(\lambda-\alpha, \alpha\hat\rho-1, \alpha\rho-1) & \frac{\sin(\alpha\pi\hat\rho)}{\sin(\alpha\pi\rho)} \Psi(\lambda-\alpha, \alpha\hat\rho-1, \alpha\rho-1) \\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat\rho)} \Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat\rho-1) & \Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat\rho-1) \end{pmatrix}.$$

Note that the function Ψ can also be written in terms of hypergeometric functions, specifically

$$\Psi(a, b, c) = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \, {}_2F_1 \begin{pmatrix} -c & a+1 \\ & a+b+2 \end{pmatrix}; -1 \end{pmatrix},$$

where ${}_2F_1$ is the usual Hypergeometric function. There are many known identities for such hypergeometric functions, see for example [1]. The appearance of hypergeometric functions is closely tied in with the fact that we are working with stable processes, for example [17, Theorem 1] describes

the laws of various conditioned stable processes in terms of what are called hypergeometric Lévy processes.

The factorisation of $\boldsymbol{F}$ first appeared in Kyprianou [20]. Here our factorisation of $\boldsymbol{F}^{-1}$ is completely independent from the derivation in [20], moreover there is no clear way to invert the factors in [20] to derive our results. The Bernstein functions that appear in [20] have not, to our knowledge, appeared in the literature and are in fact considerably harder to do computations with, whereas the factorisation that appears here is given in terms of well studied hypergeometric functions. Our proof here is much simpler and shorter as it only relies on entrance and exit probabilities of X.

Expressing the factorisation in terms of the inverse matrices has a considerable advantage in that the potential measures of the MAP are easily identified. To do, this we let $\boldsymbol{u}$ denote the unique matrix valued function so that, for $\lambda \geq 0$,

$$\int_0^\infty e^{-\lambda x} \boldsymbol{u}_{i,j}(x) dx = \boldsymbol{\kappa}_{i,j}^{-1}(\lambda) \qquad \text{for each } i, j = \pm 1.$$

Similarly, let $\hat{\boldsymbol{u}}$ denote the unique matrix valued function so that, for $\lambda \geq 0$,

$$\int_0^\infty e^{-\lambda x} \hat{\boldsymbol{u}}_{i,j}(x) dx = \hat{\boldsymbol{\kappa}}_{i,j}^{-1}(\lambda) \qquad \text{for each } i, j = \pm 1.$$

The following corollary follows from Theorem 1.4 by using the substitution $x = -\log u$ in the definition of Ψ.

Corollary 1.5. *The potential density are given by the following.*

Case $\alpha \in (0,1)$: For $x \geq 0$,

$$\boldsymbol{u}(x)$$
$$= \frac{1}{2^\alpha \Gamma(1-\alpha)} \begin{pmatrix} \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}(1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}} & \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}(1-e^{-x})^{\alpha\rho}(1+e^{-x})^{\alpha\hat{\rho}-1} \\ \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}(1-e^{-x})^{\alpha\hat{\rho}}(1+e^{-x})^{\alpha\rho-1} & \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}(1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho} \end{pmatrix}$$

and

$$\hat{\boldsymbol{u}}(x)$$
$$= \frac{1}{2^\alpha \Gamma(1-\alpha)} \begin{pmatrix} \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}(e^x-1)^{\alpha\hat{\rho}-1}(e^x+1)^{\alpha\rho} & \frac{\sin(\alpha\pi\hat{\rho})\Gamma(1-\alpha\rho)}{\sin(\alpha\pi\rho)\Gamma(\alpha\hat{\rho})}(e^x-1)^{\alpha\hat{\rho}}(e^x+1)^{\alpha\rho-1} \\ \frac{\sin(\alpha\pi\rho)\Gamma(1-\alpha\hat{\rho})}{\sin(\alpha\pi\hat{\rho})\Gamma(\alpha\rho)}(e^x-1)^{\alpha\rho}(e^x+1)^{\alpha\hat{\rho}-1} & \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}(e^x-1)^{\alpha\rho-1}(e^x+1)^{\alpha\hat{\rho}} \end{pmatrix}.$$

Case $\alpha = 1$: For $x \geq 0$,

$$\boldsymbol{u}(x) = \hat{\boldsymbol{u}}(x) = \begin{pmatrix} (1-e^{-x})^{-1/2}(1+e^{-x})^{1/2} & (1-e^{-x})^{1/2}(1+e^{-x})^{-1/2} \\ (1-e^{-x})^{1/2}(1+e^{-x})^{-1/2} & (1-e^{-x})^{-1/2}(1+e^{-x})^{1/2} \end{pmatrix}.$$

Case $\alpha \in (1,2]$: For $x \geq 0$,

$$\boldsymbol{u}(x) = \frac{\alpha-1}{2} \begin{pmatrix} (1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}} & (1-e^{-x})^{\alpha\rho}(1+e^{-x})^{\alpha\hat{\rho}-1} \\ (1-e^{-x})^{\alpha\hat{\rho}}(1+e^{-x})^{\alpha\rho-1} & (1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho} \end{pmatrix}$$
$$- \frac{(\alpha-1)^2}{2(\lambda+\alpha-1)} \begin{pmatrix} (1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}-1} & (1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}-1} \\ (1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho-1} & (1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho-1} \end{pmatrix}$$

and

$$\hat{\boldsymbol{u}}(x) = \frac{\alpha-1}{2}\begin{pmatrix} (e^x-1)^{\alpha\hat\rho-1}(e^x+1)^{\alpha\rho} & \frac{\sin(\alpha\pi\hat\rho)}{\sin(\alpha\pi\rho)}(e^x-1)^{\alpha\rho}(e^x+1)^{\alpha\rho-1} \\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat\rho)}(e^x-1)^{\alpha\rho}(e^x+1)^{\alpha\hat\rho-1} & (e^x-1)^{\alpha\rho-1}(e^x+1)^{\alpha\hat\rho} \end{pmatrix}$$
$$- \frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix} (e^x-1)^{\alpha\hat\rho-1}(e^x+1)^{\alpha\rho-1} & \frac{\sin(\alpha\pi\hat\rho)}{\sin(\alpha\pi\rho)}(e^x-1)^{\alpha\hat\rho-1}(e^x+1)^{\alpha\rho-1} \\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat\rho)}(e^x-1)^{\alpha\rho-1}(e^x+1)^{\alpha\hat\rho-1} & (e^x-1)^{\alpha\rho-1}(e^x+1)^{\alpha\hat\rho-1} \end{pmatrix}.$$

Before concluding this section, we also remark that the explicit nature of the factorisation of the Lamperti-stable MAP suggests that other factorisations of MAPs in a larger class of such processes may also exist. Indeed, following the introduction of the Lamperti-stable Lévy process in [6], for which an explicit Wiener–Hopf factorisation are available, it was quickly discovered that many other explicit Wiener–Hopf factorisations could be found by studying related positive self-similar path functionals of stable processes. In part, this stimulated the definition of the class of hypergeometric Lévy processes for which the Wiener–Hopf factorisation is explicit; see [17, 19, 18]. One might therefore also expect a general class of MAPs to exist, analogous to the class of hypergeometric Lévy processes, for which a matrix factorisation such as the one presented above, is explicitly available. Should that be the case, then the analogue of fluctuation theory for Lévy processes awaits further development in concrete form, but now for 'hypergeometric' MAPs. See for example some of the general fluctuation theory for MAPs that appears in the Appendix of [12].

1.3 Outline of the paper

The rest of the paper is structured as follows. In Section 2 we introduce some technical background material for the paper. Specifically, we introduce Markov additive processes (MAPs) in more detail, the Weiner–Hopf factorisation for MAPs and the Lamperti–Kiu transformation for real self-similar Markov processes. We prove the results by separating into three cases. In Section 3 we show Theorem 1.4 for $\alpha \in (0,1)$, and Proposition 1.1. In Section 4 we prove Theorem 1.4 for $\alpha \in (1,2)$, and Proposition 1.2. In Section 5 we show Theorem 1.4 for $\alpha = 1$. Finally in Section 6 we prove Theorem 1.3.

2 MAPs and matrix Wiener–Hopf factorisation

2.1 Markov Additive Processes

In this section we will work with a (possibly killed) Markov processes $(\xi, J) = ((\xi_t, J_t) : t \geq 0)$ on $\mathbb{R} \times \{1, -1\}$. For convenience, we will always assume that J is irreducible on $\{1, -1\}$. For such a process (ξ, J) we let $\mathbf{P}_{x,i}$ be the law of (ξ, J) started from the state (x, i).

Definition 2.1. *A Markov process (ξ, J) is called a Markov additive process (MAP) on $\mathbb{R} \times \{1, -1\}$ if, for any $t \geq 0$ and $j = -1, 1$, given $\{J_t = j\}$, the process $((\xi_{s+t} - \xi_t, J_{s+t}) : s \geq 0)$ has the same law as (ξ, J) under $\mathbf{P}_{0,j}$.*

The topic of MAPs are covered in various parts of the literature. We reference [11, 10, 2, 3, 8, 16] to name but a few of the many texts and papers. It turns out that a MAP on $\mathbb{R} \times \{1, -1\}$ requires five characteristic components: two independent and Lévy processes (possibly killed but not necessarily with the same rates), say $\chi_1 = (\chi_1(t) : t \geq 0)$ and $\chi_{-1} = (\chi_{-1}(t) : t \geq 0)$, two independent random

variables, say $\Delta_{-1,1}$ and $\Delta_{1,-1}$ on $\mathbb{R}$ and a 2×2 intensity matrix, say $\boldsymbol{Q} = (q_{i,j})_{i,j=\pm 1}$. We call the quintuple $(\chi_1, \chi_{-1}, \Delta_{-1,1}, \Delta_{1,-1}, \boldsymbol{Q})$ the driving factors of the MAP.

Definition 2.2. *A Markov additive processes on $\mathbb{R} \times \{1, -1\}$ with driving factors $(\chi_1, \chi_{-1}, \Delta_{-1,1}, \Delta_{1,-1}, \boldsymbol{Q})$ is defined as follows. Let $J = (J(t) : t \geq 0)$ be a continuous time Markov process on $\{1, -1\}$ with intensity matrix $\boldsymbol{Q}$. Let $\sigma_1, \sigma_2, \ldots$ denote the jump times of J. Set $\sigma_0 = 0$ and $\xi_0 = x$, then for $n \geq 0$ iteratively define*

$$\xi(t) = \mathbb{1}_{n>0}(\xi(\sigma_n-) + U^{(n)}_{J(\sigma_n-),J(\sigma_n)}) + \chi^{(n)}_{J(\sigma_n)}(t - \sigma_n) \qquad t \in [\sigma_n, \sigma_{n+1}),$$

where $(U^{(n)}_{i,j})_{n \geq 0}$ and $\chi^{(n)}_i$ are i.i.d. with distributions $\Delta_{i,j}$ and χ_i respectively.

It is not hard to see that the construction above results in a MAP. Conversely we have that every MAP arises in this manner, we refer to [2, XI.2a] for a proof.

Proposition 2.3. *A Markov process (ξ, J) is a Markov additive process if and only if it is there exists a quintuple of driving factors $(\chi_1, \chi_{-1}, \Delta_{-1,1}, \Delta_{1,-1}, \boldsymbol{Q})$. Consequently, every Markov additive process on $\mathbb{R} \times \{1, -1\}$ can be identified uniquely by a quintuple $(\chi_1, \chi_{-1}, \Delta_{-1,1}, \Delta_{1,-1}, \boldsymbol{Q})$ and every quintuple defines a unique Markov additive process.*

Let ψ_{-1} and ψ_1 be the Laplace exponent of χ_{-1} and χ_1 respectively (when they exist). For $z \in \mathbb{C}$, let $\boldsymbol{G}(z)$ denote the matrix whose entries are given by $\boldsymbol{G}_{i,j}(z) = \mathbf{E}[e^{z\Delta_{i,j}}]$ (when they exists), for $i \neq j$ and $\boldsymbol{G}_{i,j}(z) = 1$. For $z \in \mathbb{C}$, when it exists, define

$$\boldsymbol{F}(z) := \mathrm{diag}(\psi_1(z), \psi_{-1}(z)) - \boldsymbol{Q} \circ \boldsymbol{G}(z), \tag{7}$$

where $\mathrm{diag}(\psi_1(z), \psi_{-1}(z))$ is the diagonal matrix with entries $\psi_1(z)$ and $\psi_{-1}(z)$, and $\circ$ denotes element–wise multiplication. It is not hard to check that $\boldsymbol{F}$ is unique for each quintuple $(\chi_1, \chi_{-1}, \Delta_{-1,1}, \Delta_{1,-1}, \boldsymbol{Q})$ and furthermore for each $i, j = \pm 1$ and $t \geq 0$,

$$\mathbf{E}_{0,i}[e^{z\xi_t}; J_t = j] = (e^{\boldsymbol{F}(z)t})_{i,j}.$$

For this reason we refer to $\boldsymbol{F}$ as a MAP-exponent.

2.2 Ladder height process and the matrix Weiner–Hopf factorisation

Here we will introduce the notion of the ladder height processes for MAPs and introduce the matrix Weiner–Hopf factorisation. It may be useful for the reader to compare this to the treatment of these topics for Lévy processes in [21, Chapter 6].

Let (ξ, J) be a MAP and define the process $\bar{\xi} = (\bar{\xi}_t : t \geq 0)$ by setting $\bar{\xi}_t = \sup_{s \leq t} \xi_s$. Then it can be shown (see [15, Theorem 3.10] or [4, Chapter IV]) that there exists two non-constant increasing processes $\bar{L}^{(-1)} = (\bar{L}^{(-1)}_t : t \geq 0)$ and $\bar{L}^{(1)} = (\bar{L}^{(1)}_t : t \geq 0)$ such that $\bar{L}^{(-1)}$ increases on the closure of the set $\{t : (\xi_t, J_t) = (\bar{\xi}_t, -1)\}$ and $\bar{L}^{(1)}$ increases on the closure of the set $\{t : (\xi_t, J_t) = (\bar{\xi}_t, 1)\}$. Moreover $\bar{L}^{(-1)}$ and $\bar{L}^{(1)}$ are unique up to a constant multiples. We call $\bar{L} = \bar{L}^{(-1)} + \bar{L}^{(1)}$ the local time at the maximum.

It may be the case that $\bar{L}_\infty < \infty$, for example if ξ drifts to $\pm\infty$. In such a case, both $\bar{L}^{(-1)}_\infty$ and $\bar{L}^{(1)}_\infty$ are distributed exponentially. Since we the processes $\bar{L}^{(-1)}$ and $\bar{L}^{(1)}$ are unique up to constants, we henceforth assume that the normalisation that ensures

$$\text{both } \bar{L}^{(-1)}_\infty \text{ and } \bar{L}^{(1)}_\infty \text{ are distributed as exponentials with rate } 1. \tag{8}$$

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The ascending height processes $H^+ = (H_t^+ : t \geq 0)$ is defined as

$$(H_t^+, J_t^+) := (\bar{\xi}_{\bar{L}_t^{-1}}, J_{\bar{L}_t^{-1}}) \qquad t \in [0, \bar{L}_\infty),$$

where the inverse in the above equation is taken to be right continuous. At time $\bar{L}_\infty$ we send (H^+, J^+) to a cemetery state and declare the process killed. It is not hard to see that (H^+, J^+) is itself a MAP.

We denote by $\boldsymbol{\kappa}$ the MAP–exponent of (H^+, J^+)

and note that $\boldsymbol{\kappa}(\theta)$ is well defined for at least $\theta \geq 0$. Similarly, we define (H^-, J^-), called the descending height, by using $-\bar{\xi}$ in place of $\bar{\xi}$. We denote by $\underline{\boldsymbol{\kappa}}$ the MAP–exponent of (H^-, J^-).

For a Lévy process, its dual is simply given by its negative. The dual of a MAP is a little bit more involved. Firstly, since J is assumed to be irreducible on $\{1, -1\}$, it follows that it is reversible with respect to a unique stationary distribution $\pi = (\pi_1, \pi_{-1})$. We denote by $\hat{\boldsymbol{Q}} = (\hat{q}_{i,j})_{i,j=\pm 1}$ the matrix whose entries are given by

$$\hat{q}_{i,j} = \frac{\pi_j}{\pi_i} q_{j,i}.$$

The MAP–exponent, $\hat{\boldsymbol{F}}$, of the dual $(\hat{\xi}, \hat{J})$ is given by

$$\hat{\boldsymbol{F}}(z) = \mathrm{diag}(\psi_1(-z), \psi_{-1}(-z)) - \hat{\boldsymbol{Q}} \circ \boldsymbol{G}(-z), \tag{9}$$

whenever the right-hand side exists. Define

$$\boldsymbol{\Delta}_\pi = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_{-1} \end{pmatrix}.$$

Then the following lemma follows immediately from (9).

Lemma 2.4. *For each $z \in \mathbb{C}$,*
$$\hat{\boldsymbol{F}}(z) = \boldsymbol{\Delta}_\pi^{-1} \boldsymbol{F}(-z)^T \boldsymbol{\Delta}_\pi,$$
where $\boldsymbol{F}(-z)^T$ denotes the transpose of $\boldsymbol{F}(-z)$.

Remark 2.5. *Notice that*
$$\boldsymbol{F}(0) = \begin{pmatrix} q_{1,1} & -q_{1,1} \\ -q_{-1,-1} & q_{-1,-1} \end{pmatrix} = -\boldsymbol{Q}.$$

Hence the matrix $\boldsymbol{\Delta}_\pi$ can be computed leading to the form in (5). Also note that it is sufficient to use a constant multiple of the matrix $\boldsymbol{\Delta}_\pi$.

Similarly to how we obtained (H^+, J^+), we denote by $(\hat{H}^+, \hat{J}^+)$ the ascending ladder height process of $-X$, which is obtained from the dual process $(\hat{\xi}, \hat{J})$.

We denote by $\hat{\boldsymbol{\kappa}}$ the MAP–exponent of $(\hat{H}^+, \hat{J}^+)$.

The matrix $\hat{\boldsymbol{\kappa}}$ can also be obtained from $\underline{\boldsymbol{\kappa}}$, the MAP–exponent of the descending ladder height process, as follows,

$$\hat{\boldsymbol{\kappa}}(\theta) = \begin{pmatrix} 1 & \frac{\pi_{-1}}{\pi_1} \\ \frac{\pi_1}{\pi_{-1}} & 1 \end{pmatrix} \circ \underline{\boldsymbol{\kappa}}(\theta) \qquad \theta \in \mathbb{C} \tag{10}$$

where once again $\circ$ is for element-wise multiplication.

Now we can state the Weiner–Hopf factorisation for MAPs as it appears in (4). The proof of this theorem can be found in [16] and [15].

Theorem 2.6. *For each $\lambda \in \mathbb{R}$ we have that*

$$\boldsymbol{F}(i\lambda) = -\boldsymbol{\Delta}_\pi^{-1}\hat{\boldsymbol{\kappa}}(i\lambda)^T\Delta_\pi\boldsymbol{\kappa}(-i\lambda).$$

We complete this section by remarking that if X is an rssMp with Lamperti–Kiu exponent (ξ, J), then ξ encodes the radial distance of X and J encodes the sign of X. Consequently if (H^+, J^+) is the ascending height process of (ξ, J), then H^+ encodes the supremum of $|X|$ and J^+ encodes the sign of where the supremum was reached. Similarly if (H^-, J^-) is the descending height process of (ξ, J), then H^- encodes the infimum of $|X|$ and J^- encodes the sign of where the infimum was reached.

Although it is not so obvious, one can obtain $\hat{\boldsymbol{\kappa}}$ from $\boldsymbol{\kappa}$ as given by the following lemma which is a consequence of particular properties of the stable process. We leave the proof out (see [20, Section 7]).

Lemma 2.7. *For each $\lambda \geq 0$,*

$$\hat{\boldsymbol{\kappa}}(i\lambda) = \boldsymbol{\Delta}_\pi^{-1}\boldsymbol{\kappa}(i\lambda + 1 - \alpha)|_{\rho\leftrightarrow\hat{\rho}}\boldsymbol{\Delta}_\pi,$$

where $|_{\rho\leftrightarrow\hat{\rho}}$ indicates exchanging the roles of ρ with $\hat{\rho}$.

3 Results for $\alpha \in (0, 1)$

In this section we will prove Theorem 1.4 for $\alpha \in (0, 1)$ and Proposition 1.1.

Suppose that $(X, \mathbb{P}_x)$ is a α-stable process started at $x \neq 0$ with $\alpha < 1$ and let (ξ, J) be the MAP in the Lamperti–Kiu transformation of X. Let (H^-, J^-) be the descending height process of (ξ, J) and define $\boldsymbol{U}^-$ by

$$\boldsymbol{U}_{i,j}^-(x) = \int_0^\infty \mathbf{P}_{0,i}(H_t^- \leq x; J_t^- = j)dt.$$

Note that we set $\mathbf{P}_{0,i}(H_t^- \leq x; J_t^- = j) = 0$ if (H^-, J^-) is killed prior to time t. We will see that $x \mapsto \boldsymbol{U}_{i,j}^-(x)$ is differentiable, and we will pre-emptively denote by $\boldsymbol{u}^-$ the derivative matrix of $\boldsymbol{U}^-$. We present an auxiliary result.

Lemma 3.1. *For an α-stable process with $\alpha < 1$ we have that for all $x \geq 0$,*

$$\boldsymbol{u}^-(x) =$$

$$\frac{1}{2^\alpha\Gamma(1-\alpha)}\begin{pmatrix} \dfrac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}(e^x - 1)^{\alpha\hat{\rho}-1}(e^x + 1)^{\alpha\rho} & \dfrac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}(e^x - 1)^{\alpha\hat{\rho}}(e^x + 1)^{\alpha\rho-1} \\ \dfrac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}(e^x - 1)^{\alpha\rho}(e^x + 1)^{\alpha\hat{\rho}-1} & \dfrac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}(e^x - 1)^{\alpha\rho-1}(e^x + 1)^{\alpha\hat{\rho}} \end{pmatrix}.$$

Now we show that Theorem 1.4 follows from Lemma 3.1.

Proof of Theorem 1.4 for $\alpha < 1$. Let $\underline{\boldsymbol{\kappa}}$ be the MAP–exponent of (H^-, J^-). Then for $\lambda \geq 0$,

$$\int_0^\infty e^{-\lambda x}\boldsymbol{u}_{i,j}^-(x)dx = \int_0^\infty \mathbf{E}_{0,i}[e^{-\lambda H_t^-}; J_t^- = j]dt = \underline{\boldsymbol{\kappa}}_{i,j}^{-1}(\lambda).$$

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Thus from Lemma 3.1, we can integrate to obtain e.g., for $i, j = -1$ and $\lambda \geq 0$,

$$\underline{\boldsymbol{\kappa}}_{-1,-1}^{-1}(\lambda) = \frac{\Gamma(1 - \alpha\rho)}{2^\alpha \Gamma(1 - \alpha)\Gamma(\alpha\rho)} \int_0^\infty e^{-\lambda x}(e^x - 1)^{\alpha\rho - 1}(e^x + 1)^{\alpha\hat{\rho}} dx$$

$$= \frac{\Gamma(1 - \alpha\rho)}{2^\alpha \Gamma(1 - \alpha)\Gamma(\alpha\rho)} \int_0^1 u^{\lambda - \alpha}(1 - u)^{\alpha\rho - 1}(1 + u)^{\alpha\hat{\rho}} du,$$

where we have used the substitution $u = e^{-x}$. Once the remaining components of $\underline{\boldsymbol{\kappa}}^{-1}$ have been obtained similarly to above, we use (10) to get $\hat{\boldsymbol{\kappa}}^{-1}$ and then apply Lemma 2.7 to get $\boldsymbol{\kappa}^{-1}$. The details of the computation are left out. $\qquad\square$

We are left to prove Lemma 3.1. We will do so by first considering the process X started at $x > 0$. The case when $x < 0$ will follow by considering the dual $\hat{X} = -X$.

Recall that $\underline{m}$ is the unique time such that

$$|X_t| \geq |X_{\underline{m}}| \qquad \text{for all } t \geq 0.$$

Our proof relies on the analysis of the random variable $X_{\underline{m}}$. Notice that $X_{\underline{m}}$ may be positive or negative and takes values in $[-x, x]$.

Before we derive the law of $X_{\underline{m}}$, we first present the following lemma which appears in [22, Corollary 1.2].

Lemma 3.2. *Let $\tau^{(-1,1)} := \inf\{t \geq 0 : |X_t| < 1\}$. We have that, for $x > 1$,*

$$\mathbb{P}_x(\tau^{(-1,1)} = \infty) = \Phi(x),$$

where

$$\Phi(x) = \frac{\Gamma(1 - \alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha)} \int_0^{(x-1)/(x+1)} t^{\alpha\hat{\rho} - 1}(1 - t)^{-\alpha} dt.$$

Lemma 3.2 immediately gives that the law of $|X_{\underline{m}}|$ as

$$\mathbb{P}_x(|X_{\underline{m}}| > z) = \Phi(x/z), \qquad \text{for } z \in [0, x].$$

Indeed, the event $\{|X_{\underline{m}}| > z\}$ occurs if and only if $\tau^{(-z,z)} = \infty$. From the scaling property of X we get that $\mathbb{P}_x(\tau^{(-z,z)} = \infty) = \mathbb{P}_{x/z}(\tau^{(-1,1)} = \infty) = \Phi(x/z)$.

We first begin to derive the law of $X_{\underline{m}}$ which shows Proposition 1.1.

Proof of Proposition 1.1. Fix $x > 0$. Similarly to the definition of $\underline{m}$, we define $\underline{m}^+$ and $\underline{m}^-$ as follows: Let $\underline{m}^+$ be the unique time such that $X_{\underline{m}^+} > 0$ and

$$X_t \geq X_{\underline{m}^+} \qquad \text{for all } t \geq 0 \text{ such that } X_t > 0.$$

Similarly let $\underline{m}^-$ be the unique time such that $X_{\underline{m}^-} < 0$ and

$$X_t \leq X_{\underline{m}^-} \qquad \text{for all } t \geq 0 \text{ such that } X_t < 0.$$

In words, $\underline{m}^+$ and $\underline{m}^-$ are the times when X is at the closest point to the origin on the positive and negative side of the origin, respectively. Consequently, we have that $X_{\underline{m}} > 0$ if and only if $X_{\underline{m}^+} < |X_{\underline{m}^-}|$. We now have that

$$\mathbb{P}(|X_{\underline{m}^-}| > u; X_{\underline{m}^+} > v) = \mathbb{P}_x(\tau^{(-u,v)} = \infty) = \Phi\left(\frac{2x + u - v}{u + v}\right),$$

where Φ is defined in Lemma 3.2 and in the final equality we have scaled space and used the self-similarity of X.

Next we have that for $z \geq 0$,

$$\frac{\mathbb{P}_x(X_{\underline{m}} \in dz)}{dz} = -\frac{\partial}{\partial v}\mathbb{P}(|X_{\underline{m}^-}| > z; X_{\underline{m}^+} > v)|_{v=z} = -\frac{\partial}{\partial v}\Phi\left(\frac{2x + z - v}{z + v}\right)|_{v=z}$$
$$= \frac{x + z}{2z^2}\Phi'\left(\frac{x}{z}\right). \tag{11}$$

The proposition for $z > 0$ now follows from an easy computation. The result for $z < 0$ follows similarly. $\qquad\square$

Now we will use (11) to show Lemma 3.1. We will need the following simple lemma which appears in the Appendix of [12].

Lemma 3.3. *Let* $T_0^- := \inf\{t \geq 0 : \xi_t < 0\}$ *then for* $i, j = -1, 1$ *and* $y > 0$,

$$\mathbf{P}_{y,i}(T_0^- = \infty; J_{\varphi(\underline{m})} = j) = \boldsymbol{U}_{i,j}^-(y).$$

Let us now describe the event $\{T_0^- = \infty; J_{\varphi(\underline{m})} = j\}$ in terms of the underlying process X. The event $\{T_0^- = \infty; J_{\varphi(\underline{m})} = 1\}$ occurs if and only if $\tau^{(-1,1)} = \infty$ and furthermore the point at which X is closest to the origin is positive, i.e. $X_{\underline{m}} > 0$. Thus $\{T_0^- = \infty; J_{\varphi(\underline{m})} = 1\}$ occurs if and only if $X_{\underline{m}} > 1$. Using Lemma 3.3 and (11) we have that

$$\boldsymbol{U}_{1,1}^-(x) = \mathbf{P}_{x,i}(T_0^+ = \infty; J_{\varphi(\underline{m})} = j) = \mathbb{P}_{e^x}(X_{\underline{m}} > 1)$$
$$= \frac{1}{2}\int_1^{e^x}(e^x + z)\Phi'\left(\frac{e^x}{z}\right)z^{-2}dz$$
$$= \frac{1}{2}\int_1^{e^x}(1 + 1/u)\Phi'(u)du,$$

where in the final equality we have used the substitution $u = e^x/z$. Differentiating the above equation we get that

$$\boldsymbol{u}_{1,1}^-(x) = \frac{1}{2}(e^x + 1)\Phi'(e^x) = \frac{\Gamma(1 - \alpha\rho)}{2^\alpha\Gamma(1 - \alpha)\Gamma(\alpha\hat{\rho})}(e^x - 1)^{\alpha\hat{\rho}-1}(e^x + 1)^{\alpha\rho}. \tag{12}$$

Similarly considering the event $\{T_0^+ = \infty; J_{\varphi(\underline{m})} = -1\}$ we get that

$$\boldsymbol{u}_{1,-1}^-(x) = \frac{1}{2}(e^x - 1)\Phi'(e^x) = \frac{\Gamma(1 - \alpha\rho)}{2^\alpha\Gamma(1 - \alpha)\Gamma(\alpha\hat{\rho})}(e^x - 1)^{\alpha\hat{\rho}}(e^x + 1)^{\alpha\rho-1}.$$

Notice now that $\boldsymbol{u}_{1,j}^-$ only depends on α and ρ. Consider now the dual process $\hat{X} = (-X_t : t \geq 0)$. This process is the same as X albeit $\rho \leftrightarrow \hat{\rho}$. To derive the row $\boldsymbol{u}_{-1,j}^-$ we can use $\hat{X}$ in the computations above and this implies that $\boldsymbol{u}_{-1,j}^-$ is the same as $\boldsymbol{u}_{1,-j}^-$ but exchanging the roles of ρ with $\hat{\rho}$. This concludes the proof of Lemma 3.1.

4 Proof of Theorem 1.4 for $\alpha \in (1,2)$

In this section we will prove Theorem 1.4 for $\alpha \in (1,2)$. Let X be an α-stable process with $\alpha \in (1,2]$ and let (ξ, J) be the MAP associated to X via the Lamperti–Kiu transformation. The proof here is very similar to that of the case when $\alpha < 1$, thus we skip some of the details.

Since $\alpha \in (1,2]$ we have that $\tau^{\{0\}} := \inf\{t \geq 0 : X_t = 0\} < \infty$ and $X_{\tau^{\{0\}}-} = 0$ almost surely. Hence it must be the case thar ξ drifts to $-\infty$. Recall that $\overline{m}$ as the unique time for which $\overline{m} < \tau^{\{0\}}$ and

$$|X_{\overline{m}}| \geq |X_t| \qquad \text{for all } t < \tau^{\{0\}}.$$

The quantity we are interested in is $X_{\overline{m}}$. We begin with the following lemma, which is lifted from [25, Corollary 1] and also can be derived from the potential given in [23, Theorem 1].

Lemma 4.1. *For every $x \in (0,1)$ and $y \in (x,1)$,*

$$\mathbb{P}_x(\tau^{\{y\}} < \tau^{(-1,1)^c}) = (\alpha - 1)\left(\frac{x-y}{1-y^2}\right)^{\alpha-1} \bar{\Phi}\left(\left|\frac{1-xy}{x-y}\right|\right),$$

where

$$\bar{\Phi}(z) = \int_1^z (t-1)^{\alpha\rho-1}(t+1)^{\alpha\hat{\rho}-1}dt.$$

Next we prove Proposition 1.2 by expressing exit probabilities in terms of $\bar{\Phi}$. In the spirit of the proof of Proposition 1.1, we apply a linear spatial transformation to the probability $\mathbb{P}_x(\tau^{(-u,v)^c} < \tau^{\{0\}})$ and write it in terms of $\bar{\Phi}$.

Proof of Proposition 1.2. Similar to the derivation of (11) in the proof of Proposition 1.1, for each $x > 0$ and $|z| > x$,

$$\frac{\mathbb{P}_x(X_{\overline{m}} \in dz)}{dz} = \frac{\alpha - 1}{2x^{2-\alpha}|z|^\alpha}\left(|x+z|\bar{\Phi}'\left(\frac{|z|}{x}\right) - (\alpha-1)x\bar{\Phi}\left(\frac{|z|}{x}\right)\right). \tag{13}$$

The result now follows from straight forward computations. $\qquad\square$

Again we introduce the following lemma from the Appendix of [12].

Lemma 4.2. *Let $T_0^+ := \inf\{t \geq 0 : \xi > 0\}$ then for $i,j = -1,1$ and $y > 0$,*

$$\mathbf{P}_{-y,i}(T_0^+ = \infty; J_{\varphi(\overline{m})} = j) = \boldsymbol{U}_{i,j}(y).$$

Similar to the derivation in (12), we use (13) and Lemma 4.2 to get that

$$
\begin{aligned}
\boldsymbol{u}_{1,1}(x) &= \frac{d}{dx}\mathbb{P}_{e^{-x}}(X_{\overline{m}} \in (e^{-x}, 1)) \\
&= \frac{\alpha - 1}{2}\frac{d}{dx}\int_{e^{-x}}^1 dz \frac{1}{z^\alpha}e^{(2-\alpha)x}\left((e^{-x} + z)\bar{\Phi}'\left(\frac{z}{e^{-x}}\right) - (\alpha-1)e^{-x}\bar{\Phi}\left(\frac{z}{e^{-x}}\right)\right) \\
&= \frac{\alpha - 1}{2}\frac{d}{dx}\int_1^{e^x} du \frac{1}{u^\alpha}\left((1+u)\bar{\Phi}'(u) - (\alpha-1)\bar{\Phi}(u)\right) \\
&= \frac{\alpha - 1}{2}e^{-(\alpha-1)x}\left((1+e^x)\bar{\Phi}'(e^x) - (\alpha-1)\bar{\Phi}(e^x)\right) \\
&= \frac{\alpha - 1}{2}e^{-(\alpha-1)x}\left((e^x - 1)^{\alpha\rho-1}(e^x + 1)^{\alpha\hat{\rho}} - (\alpha-1)\bar{\Phi}(e^x)\right), \tag{14}
\end{aligned}
$$

where in the third equality we have used the substitution $u = z/e^{-x}$. Now we will take the Laplace transform of $\boldsymbol{u}_{1,1}$. The transform of the $\bar{\bar{\Phi}}$ term is dealt with the following lemma. The proof follows from integration by parts which we leave out.

Lemma 4.3. *Suppose that $\gamma > \alpha - 1$, then*

$$\int_0^\infty e^{-\gamma x}\bar{\bar{\Phi}}(e^x)dx = \frac{1}{\gamma}\Psi(\gamma - \alpha, \alpha\rho - 1, \alpha\hat{\rho} - 1).$$

Next we have

$$\int_0^\infty e^{-(\lambda+\alpha-1)x}(e^x - 1)^{\alpha\rho-1}(e^x + 1)^{\alpha\hat{\rho}}dx = \int_0^1 u^{\lambda-1}(1-u)^{\alpha\rho-1}(1+u)^{\alpha\hat{\rho}}du = \Psi(\lambda - 1, \alpha\rho - 1, \alpha\hat{\rho}),$$

where we have used the substitution $u = e^{-x}$. Integrating (14) and using the above equation together with Lemma 4.3 we get that

$$\boldsymbol{\kappa}_{1,1}(\lambda) = \frac{\alpha - 1}{2}\Psi(\lambda - 1, \alpha\rho - 1, \alpha\hat{\rho}) - \frac{(\alpha - 1)^2}{2(\lambda + \alpha - 1)}\Psi(\lambda - 1, \alpha\rho - 1, \alpha\hat{\rho} - 1).$$

Similar proofs give $\boldsymbol{\kappa}_{i,j}^{-1}$ for the remaining i, j. The formula for $\hat{\boldsymbol{\kappa}}^{-1}$ follows from Lemma 2.7.

5   Proof of Theorem 1.4 for $\alpha = 1$

In the case when $\alpha = 1$, the process X is a Cauchy process, which has the property that $\limsup_{t\to\infty} |X_t| = \infty$ and $\liminf_{t\to\infty} |X_t| = 0$. This means that the map (ξ, J) oscillates and hence the global minimum and maximum both do not exist so that the previous methods cannot be used. Instead we focus on a two sided exit problem as an alternative approach. (Note, the method we are about to describe also works for the other cases of α, however it is lengthy and we do not obtain the new identities *en route* in a straightforward manner as we did in Proposition 1.1 and Proposition 1.2.)

The following result follows from the compensation formula and the proof of it is identical to the case for Lévy processes, see [4, III Proposition 2] and [21, Theorem 5.8].

Lemma 5.1. *Let (H^+, J^+) be the height process of (ξ, J). For any $x > 0$ define $T_x := \inf\{t > 0 : H_t^+ > x\}$, then for any $x > 0$ and $i = \pm 1$,*

$$\mathbf{P}_{0,i}(x - H_{T_x-}^+ \in du; H_{T_x}^+ - x \in dy; J_{T_x-}^+ = 1; J_{T_x}^+ = 1) = \boldsymbol{u}_{i,1}(x - u)\pi(dy + u)du,$$

where π is some σ–finite measure on $[0, \infty)$.

Next we will calculate the over and under shoots in Lemma 5.1 by using the underlying process X. This is done in the following lemma.

Lemma 5.2. *Let $\tau_1^+ := \inf\{t \geq 0 : X_t > 1\}$ and $\tau_{-1}^- := \inf\{t \geq 0 : X_t < -1\}$. Then for $x \in (-1, 1)$, $u \in [0, (1 - x) \vee 1$ and $y \geq 0$,*

$$\mathbb{P}_x(1 - \bar{X}_{\tau_1^+-} \in du; X_{\tau_1^+} - 1 \in dy; X_{\tau_1^+-} > 0; \tau_1^+ < \tau_{-1}^-) = \frac{(1 - u + x)^{1/2}}{(1 - u - x)^{1/2}}\frac{(u + y)^{3/2}}{(2 - u + y)^{1/2}}dudy.$$

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Proof. First [23, Corollary 3] gives that for $z \in (0,1)$, $u \in [0, 1-z)$ and $v \in (u, 1]$,

$$\mathbb{P}_z(1 - \bar{X}_{\tau_1^+-} \in du; 1 - X_{\tau_1^+-} \in dv; X_{\tau_1^+} - 1 \in dy; \tau_1^+ < \tau_0^-)$$

$$= \frac{1}{\pi} \frac{x^{1/2}(1-v)^{1/2}}{(1-u-x)^{1/2}(v-u)^{1/2}(1-u)(v+y)^2} du\,dy,$$

where $\tau_0^- := \inf\{t \geq 0 : X_t < 0\}$. We wish to integrate v out of the above equation. To do this, we make the otherwise subtle observation that

$$\int_u^1 dv\,(1-v)^{1/2}(v-u)^{-1/2}(v+y)^{-2} = (u+y)^{-2}(1-u) \int_0^1 dz\,(1-z)^{1/2}z^{-1/2}\left(1 + z\frac{1-u}{u+y}\right)^{-2}$$

$$= (u+y)^{-2}(1-u)\frac{\pi}{2}\,{}_2F_1\left(2, 1/2, 2; -\frac{1-u}{u+y}\right)$$

$$= \frac{\pi}{2}(u+y)^{-3/2}(1-u)(1+y)^{-1/2},$$

where in the first equality we have used the substitution $z = (v-u)/(1-u)$. In the second equality we have used [1, Theorem 2.2.1] and the final equality follows from the Euler–transformation [1, Theorem 2.2.5].

Hence, for $x \leq 1$, $u \in [0, 1-z)$ and $y \geq 0$,

$$\mathbb{P}_z(1 - \bar{X}_{\tau_1^+-} \in du; X_{\tau_1^+} - 1 \in dy; \tau_1^+ < \tau_0^-)$$

$$= \frac{1}{2} \frac{x^{1/2}}{(1-u-x)^{1/2}(1+y)^{1/2}(u+y)^{3/2}} du\,dy. \tag{15}$$

Next we have that for $u \in [0, (1-x) \vee 1)$ and $y \geq 0$,

$$\frac{d}{du}\frac{d}{dy}\mathbb{P}_x(1 - \bar{X}_{\tau_1^+-} \in du; X_{\tau_1^+} - 1 \in dy; X_{\tau_1^+-} > 0; \tau_1^+ < \tau_{-1}^-)$$

$$= \frac{d}{dv}\frac{d}{dy}\mathbb{P}_x(1 - \bar{X}_{\tau_1^+-} \leq v; X_{\tau_1^+} - 1 \leq y; \tau_1^+ < \tau_{u-1}^-)|_{v=u}$$

$$= \frac{d}{dv}\frac{d}{dy}\mathbb{P}_{\frac{x+1-u}{2-u}}\left(1 - \bar{X}_{\tau_1^+-} \leq \frac{v}{2-u}; X_{\tau_1^+} - 1 \leq \frac{y}{2-u}; \tau_1^+ < \tau_0^-\right)|_{v=u}$$

$$= \frac{1}{2}(2-u)^{-2} \frac{\left(\frac{x+1-u}{2-u}\right)^{1/2}}{\left(1 - \frac{u}{2-u} - \frac{x+1-u}{2-u}\right)^{1/2}\left(1 + \frac{y}{2-u}\right)^{1/2}\left(\frac{u+y}{2-u}\right)^{3/2}}$$

$$= \frac{(1-u+x)^{1/2}}{(1-u-x)^{1/2}} \frac{1}{(2-u+y)^{1/2}(u+y)^{3/2}},$$

where in the second equality we have used the scaling property of X and in the third equality we have used (15). $\square$

Notice now that for each $x \geq 0$ and $i = \pm 1$,

$$\frac{d}{du}\frac{d}{dy}\mathbf{P}_{0,i}(x - H_{T_x^-}^+ \leq u; H_{T_x}^+ - x \leq y; J_{T_x^-}^+ = 1; J_{T_x}^+ = 1)$$

$$= \frac{d}{du}\frac{d}{dy}\mathbb{P}_i(\bar{X}_{\tau_{e^x}^+ -} \geq e^{x-u}; X_{\tau_{e^x}^+} \leq e^{y+x}; X_{\tau_{e^x}^+ -} > 0; \tau_{e^x}^+ < \tau_{-e^x}^-)$$

$$= \frac{d}{du}\frac{d}{dy}\mathbb{P}_{ie^{-x}}(\bar{X}_{\tau_1^+ -} \geq e^{-u}; X_{\tau_1^+} \leq e^y; X_{\tau_1^+ -} > 0; \tau_1^+ < \tau_{-1}^-)$$

$$= e^{y-u}\frac{(e^{-u} + ie^{-x})^{1/2}}{(e^{-u} - ie^{-x})^{1/2}}\frac{1}{(e^y + e^{-u})^{1/2}(e^y - e^{-u})^{3/2}},$$

where in the second equality we have used the scaling property of X and in the final equality we applied Lemma 5.2. The above equation together with Lemma 5.1 gives that for $x \geq 0$,

$$\frac{\boldsymbol{u}_{1,1}(x)}{\boldsymbol{u}_{-1,1}(x)} = \frac{1 + e^{-x}}{1 - e^{-x}}. \tag{16}$$

Next we claim that

$$\sum_{i=\pm 1} \boldsymbol{u}_{1,i}(x) = (1 - e^{-x})^{-1/2}(1 + e^{-x})^{1/2} + (1 - e^{-x})^{1/2}(1 + e^{-x})^{-1/2}. \tag{17}$$

This follows from existing literature on the Lamperti transform of the Cauchy process and we briefly describe how to verify this. Observe that $(|X_t| : t \geq 0)$ is a positive self-similar Markov process and thus can be expressed as the time–changed exponential of a Lévy process, say $\chi = (\chi_t : t \geq 0)$, see for example [21, Chapter 13]. The sum on the left-hand side of (17) is precisely the potential of the ladder height process of χ. We can verify that the potential of the height process of χ has the form given by the right-hand side of (17) as follows. The characteristic exponent of the height process of χ is given in [7, Remark 2]. Then the identity in (17) can be verified by checking that, up to a multiplicative constant, the Laplace exponent of the right-hand side matches with that in [7, Remark 2].

Now we can finish the proof. Notice first that the Cauchy process is symmetric, thus $\boldsymbol{u}_{i,j} = \boldsymbol{u}_{-i,-j}$ for each $i, j \in \{1, -1\}$. Thus from (17) we get

$$\sum_{i=\pm 1} \boldsymbol{u}_{i,1}(x) = (1 - e^{-x})^{-1/2}(1 + e^{-x})^{1/2} + (1 - e^{-x})^{1/2}(1 + e^{-x})^{-1/2}. \tag{18}$$

Solving the simultaneous equations (16) and (18) together with the fact $\boldsymbol{u}_{i,j} = \boldsymbol{u}_{-i,-j}$ gives the result for $\boldsymbol{u}$. To obtain $\hat{\boldsymbol{u}}$ we note that the reciprocal process $\widetilde{X} := 1/X_{\theta_t}$, $t \geq 0$ has the law of a Cauchy process, where $\theta_t = \inf\{s > 0 : \int_0^s |X_u|^{-2}du > t\}$ (see [5, Theorem 1]). Theorem 4 in [20] also shows that $\widetilde{X}$ has an underlying MAP which is the dual of the MAP underlying X. It therefore follows that $\hat{\boldsymbol{u}} = \boldsymbol{u}$. This finishes the proof.

Remark 5.3. *Notice that we also get the jump measure π appearing in Lemma 5.1 as*

$$\pi(dy) = \frac{e^y}{(e^y + 1)^{1/2}(e^y - 1)^{3/2}}\, dy.$$

Up to a multiplicative constant, this can also be seen in [20, equation (14)].

6   Proof of Theorem 1.3

Recall that (R, M) is a Markov process. Since R takes values on $[-1, 1]$ and (ξ, J) does not exhibit lattice behaviour, it must have a stationary distribution which does not depend on the initial position of R. For $x, y \in [-1, 1]$ and $j = \pm 1$, when it exists, define

$$\mu_j(dy) := \lim_{t \to \infty} \mathbb{P}_x(|R_t| \in dy; \, \mathrm{sgn}(R_t) = j).$$

Notice that the stationary distribution μ is given by $\mu = \mu_1 + \mu_{-1}$ (here we are pre-emptively assuming that each of the two measures on the right-hand side are absolutely continuous with respect to Lebesgue measure and so there is no 'double counting' at zero) and hence it suffices to establish an identify for μ_j.

Using duality we have for $i, j \in \pm 1$,

$$\mathbf{E}_{0,i}\left[e^{-\beta(\bar{\xi}_{\mathrm{e}_q} - \xi_{\mathrm{e}_q})}; \, J_{\mathrm{e}_q} = j\right] = \sum_{k=\pm 1} \mathbf{E}_{0,i}\left[e^{-\beta(\bar{\xi}_{\mathrm{e}_q} - \xi_{\mathrm{e}_q})}; \, J_{\overline{m}_{\mathrm{e}_q}} = k, \, J_{\mathrm{e}_q} = j\right],$$

where e_q is an independent and exponentially distributed random variable with rate q and $\overline{m}_{\mathrm{e}_q}$ is the unique time at which ξ obtains its maximum on the time interval $[0, \mathrm{e}_q]$. Appealing to the computations in the Appendix of [12], specifically equation (22) and Theorem 23, we can develop the right-hand side above so that

$$\begin{aligned}
\int_0^1 y^\beta \mu_j(dy) &= \lim_{q \downarrow 0} \mathbf{E}_{0,i}\left[e^{-\beta(\bar{\xi}_{\mathrm{e}_q} - \xi_{\mathrm{e}_q})}; \, J_{\mathrm{e}_q} = j\right] \\
&= \lim_{q \downarrow 0} \sum_{k=\pm 1} \mathbf{E}_{0,i}\left[e^{-\beta(\bar{\xi}_{\mathrm{e}_q} - \xi_{\mathrm{e}_q})}; \, J_{\overline{m}_{\mathrm{e}_q}} = k, \, J_{\mathrm{e}_q} = j\right] \\
&= \lim_{q \downarrow 0} \sum_{k=\pm 1} \mathbf{P}_{0,i}\left(J_{\overline{m}_{\mathrm{e}_q}} = k\right) \hat{\mathbf{E}}_{0,j}\left[e^{-\beta \bar{\xi}_{\mathrm{e}_q}}; \, J_{\overline{m}_{\mathrm{e}_q}} = k\right] \frac{\pi_j}{\pi_k} \\
&= \sum_{k=\pm 1} \bar{\hat{\eta}}_k(\zeta = \infty) \pi_j [\hat{\boldsymbol{\kappa}}(\beta)^{-1}]_{j,k},
\end{aligned}$$

where, for $j = \pm 1$, $\bar{\hat{\eta}}_j$ is the excursion measure of $(\bar{\xi} - \xi, J)$ from $(0, j)$ under $\hat{\mathbf{P}}_{0,j}$. Note, we have also used the fact that, $\overline{m}_{\mathrm{e}_q}$ converges to $+\infty$ almost surely as $q \to \infty$ on account of the fact that $\limsup_{t \to \infty} |X_t| = \infty$. Recall, moreovoer, that we have normalised local time such that $\bar{\hat{\eta}}_k(\zeta = \infty) = 1$.

Since $[\hat{\boldsymbol{\kappa}}(\beta)^{-1}]_{j,k}$ is the Laplace transform of $\hat{\boldsymbol{u}}_{j,k}$, it now follows that

$$\left.\frac{d\mu_j}{dy}(y)\right|_{y=e^{-x}} e^{-x} = \pi_j \sum_{k=\pm 1} \hat{\boldsymbol{u}}_{j,k}(x), \qquad x \geq 0.$$

Said another way,

$$\mu_j(dy) = \frac{\pi_j}{y} \sum_{k=\pm 1} \hat{\boldsymbol{u}}_{j,k}(-\log y) dy, \qquad y \in [0, 1].$$

The proof is completed by importing the specific expressions for the quantities in the right-hand side above, in particular from (5) and Corollary 1.5. $\qquad\square$

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Acknowledgements

References

[1] George E. Andrews, Richard Askey, and Ranjan Roy. *Special functions*. Vol. 71. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1999, pp. xvi+664.

[2] Søren Asmussen. *Applied probability and queues*. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons, Ltd., Chichester, 1987, pp. x+318.

[3] Søren Asmussen and Hansjörg Albrecher. *Ruin probabilities*. Second. Advanced Series on Statistical Science & Applied Probability, 14. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010, pp. xviii+602.

[4] Jean Bertoin. *Lévy processes*. Vol. 121. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1996, pp. x+265.

[5] K. Bogdan and T. Żak. "On Kelvin transformation". In: *J. Theoret. Probab.* 19.1 (2006), pp. 89–120.

[6] M. E. Caballero and L. Chaumont. "Conditioned stable Lévy processes and the Lamperti representation". In: *J. Appl. Probab.* 43.4 (2006), pp. 967–983.

[7] M. E. Caballero, J. C. Pardo, and J. L. Pérez. "Explicit identities for Lévy processes associated to symmetric stable processes". In: *Bernoulli* 17.1 (2011), pp. 34–59.

[8] Loïc Chaumont, Henry Pantí, and Víctor Rivero. "The Lamperti representation of real-valued self-similar Markov processes". In: *Bernoulli* 19.5B (2013), pp. 2494–2523.

[9] Loïc Chaumont et al. "Fluctuation theory and exit systems for positive self-similar Markov processes". In: *Ann. Probab.* 40.1 (2012), pp. 245–279.

[10] Erhan Çinlar. "Lévy systems of Markov additive processes". In: *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 31 (1974/75), pp. 175–185.

[11] Erhan Çinlar. "Markov additive processes. I, II". In: *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 24 (1972), 85–93; ibid. 24 (1972), 95–121.

[12] Steffen Dereich, Leif Döring, and Andreas E. Kyprianou. "Real Self-Similar Processes Started from the Origin". In: *arXiv preprint arXiv:1501.00647* (2015).

[13] R. A. Doney and A. E. Kyprianou. "Overshoots and undershoots of Lévy processes". In: *Ann. Appl. Probab.* 16.1 (2006), pp. 91–106.

[14] S. Fourati. "Points de croissance des processus de Lévy et théorie générale des processus". In: *Probab. Theory Related Fields* 110.1 (1998), pp. 13–49.

[15] H. Kaspi. "On the symmetric Wiener-Hopf factorization for Markov additive processes". In: *Z. Wahrsch. Verw. Gebiete* 59.2 (1982), pp. 179–196.

[16] Przemysław Klusik and Zbigniew Palmowski. "A note on Wiener-Hopf factorization for Markov additive processes". In: *J. Theoret. Probab.* 27.1 (2014), pp. 202–219.

[17] A. Kuznetsov and J. C. Pardo. "Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes". In: *Acta Appl. Math.* 123 (2013), pp. 113–139.

[18] A. Kuznetsov et al. "A Wiener-Hopf Monte Carlo simulation technique for Lévy processes". In: *Ann. Appl. Probab.* 21.6 (2011), pp. 2171–2190.

[19] A. E. Kyprianou, J. C. Pardo, and A. R. Watson. "The extended hypergeometric class of Lévy processes". In: *J. Appl. Probab.* 51A.Celebrating 50 Years of The Applied Probability Trust (2014), pp. 391–408.

[20] Andreas E. Kyprianou. "Deep factorisation of the stable process". In: *arXiv preprint arXiv:1502.07399* (2015).

[21] Andreas E. Kyprianou. *Fluctuations of Lévy processes with applications*. Second. Universitext. Introductory lectures. Springer, Heidelberg, 2014, pp. xviii+455.

[22] Andreas E. Kyprianou, Juan Carlos Pardo, and Alexander R. Watson. "Hitting distributions of α-stable processes via path censoring and self-similarity". In: *Ann. Probab.* 42.1 (2014), pp. 398–430.

[23] Andreas E. Kyprianou and Alexander R. Watson. "Potentials of stable processes". In: *Séminaire de Probabilités XLVI*. Springer, 2014, pp. 333–343.

[24] John Lamperti. "Semi-stable Markov processes. I". In: *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 22 (1972), pp. 205–225.

[25] Christophe Profeta and Thomas Simon. "On the harmonic measure of stable processes". In: *arXiv preprint arXiv:1501.03926* (2015).

[26] René L. Schilling, Renming Song, and Zoran Vondraček. *Bernstein functions*. Second. Vol. 37. de Gruyter Studies in Mathematics. Theory and applications. Walter de Gruyter & Co., Berlin, 2012, pp. xiv+410.

[27] Lajos Takács. *Combinatorial methods in the theory of stochastic processes*. John Wiley & Sons, Inc., New York-London-Sydney, 1967, pp. xi+262.