ENTRANCE LAWS AT THE ORIGIN OF SELF-SIMILAR MARKOV PROCESSES IN HIGH DIMENSIONS

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Abstract In this paper we consider the problem of finding entrance laws at the origin for self-similar Markov processes in $\mathbb{R}^d$, killed upon hitting the origin. Under mild assumptions, we show the existence of an entrance law and the convergence to this law when the process is started close to the origin. We obtain an explicit description of the process started from the origin as the time reversal of the original self-similar Markov process conditioned to hit the origin.

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Part I

Entrance laws of ssMp

1. Introduction. Suppose $\mathcal{H}$ is a locally compact subspace of $\mathbb{R}^d \setminus \{0\}$ $(d \geq 1)$. An $\mathcal{H}$-valued self-similar Markov process (ssMp for short) $(X, \mathbb{P}) = ((X_t)_{t\geq 0}, \{\mathbb{P}_z : z \in \mathcal{H}\})$ is an $\mathcal{H}$-valued càdlàg Markov process killed at 0 with $\mathbb{P}_z(X_0 = z) = 1$, which fulfils the scaling property, namely, there exists an $\alpha > 0$ such that for any $c > 0$,

$$((cX_{c^{-\alpha}t})_{t\geq 0}, \mathbb{P}_z) \text{ has the same law as } ((X_t)_{t\geq 0}, \mathbb{P}_{cz}) \quad \forall z \in \mathcal{H}.$$

It follows from the scaling property that $\mathcal{H} = c\mathcal{H}$ for all $c > 0$. Therefore $\mathcal{H}$ is necessarily a cone of $\mathbb{R}^d \setminus \{0\}$ which has the form

$$\mathcal{H} = \phi(\mathbb{R} \times \mathcal{S})$$

where $\mathcal{S}$ is a locally compact subspace of $\mathbb{S}^{d-1}$ and ϕ is the homeomorphism from $\mathbb{R} \times \mathbb{S}^{d-1}$ to $\mathbb{R}^d \setminus \{0\}$ defined by $\phi(y, \theta) = \theta \mathrm{e}^y$.

The crucial tool in the study of ssMp is the Lamperti-Kiu transform which we now describe. Suppose first that $(X, \mathbb{P}_z)$ is an $\mathcal{H}$-valued ssMp started at $z \in \mathcal{H}$ with index $\alpha > 0$ and lifetime ζ, then there exists a Markov additive process (MAP for short, see Section 2 for a rigorous definition) (ξ, Θ) on $\mathbb{R} \times \mathcal{S}$ started at $(\log \|z\|, \arg z)$ with lifetime ζ_p such that

$$X_t = \exp\{\xi_{\varphi(t)}\}\Theta_{\varphi(t)} \mathbb{1}_{\{t<\zeta\}} \qquad \forall t \geq 0, \tag{1.1}$$

where $\varphi(t)$ is the time-change defined by

$$\varphi(t) := \inf\left\{ s > 0 : \int_0^s \exp\{\alpha \xi_u\}\, \mathrm{d}u > t \right\}, \tag{1.2}$$

and $\zeta_p = \int_0^\zeta \|X_s\|^{-\alpha} \mathrm{d}s$. We denote the law of (ξ, Θ) started from $(y, \theta) \in \mathbb{R} \times \mathcal{S}$ by $\mathbf{P}_{y,\theta}$. Conversely given a MAP (ξ, Θ) under $\mathbf{P}_{y,\theta}$ with lifetime ζ_p, the process X defined by (1.1) is a ssMp started from $z = \theta \mathrm{e}^y$ with lifetime $\zeta = \int_0^{\zeta_p} \mathrm{e}^{\alpha \xi_s} \mathrm{d}s$. Roughly speaking, a MAP is a natural extension of a Lévy process in the sense that Θ is an arbitrary well behaved Markov processes and $((\xi_t, \Theta_t)_{t\geq 0}, \mathbf{P}_{x,\theta})$ is equal in law to $((\xi_t + x, \Theta_t)_{t\geq 0}, \mathbf{P}_{0,\theta})$ for all $x \in \mathbb{R}$ and $\theta \in \mathcal{S}$. Whilst MAPs have found a prominent role in e.g. classical applied probability models for queues and dams, c.f. [4] when Θ is a Markov chain, the case that Θ is a general Markov process has received somewhat less attention. Nonetheless a core base of literature exists in the general setting from the 1970s and 1980s thanks to e.g. [14, 15, 23, 24].

We denote $\mathcal{H} \cup \{0\}$ by $\mathcal{H}_0$. In this paper we look for entrance laws of ssMp at the origin, that is, the existence of a probability measure $\mathbb{P}_0$ such that the extension of $(X, \{\mathbb{P}_z : z \in \mathcal{H}_0\})$ is self-similar and in particular $\mathbb{P}_0 = \text{w-}\lim_{\mathcal{H}\ni z\to 0} \mathbb{P}_z$ in the Skorokhod topology. In Theorem 6.1 we will prove a general result with as weak assumptions as our study of the underlying MAPs permits. However, the statement of this theorem comes relatively late in this paper on account of the large amount of fluctuation theory we must first develop for general MAPs in order that the sufficient conditions to make sense. It is quite natural to expect that conditions for the existence of an entrance law will be highly non-trivial as the process Θ could essentially take on any role as a regular Markov process. Nonetheless, we want to give a flavour of the main results. We give immediately below the collection of conclusions we are aiming towards, i.e. (C1)-(C5), without addressing the technical assumptions.

The first two conclusions (C1) and (C2) seem rather specialist and pertain to analogues of classical fluctuation results for Lévy processes, but now in the setting of MAPs. However they hold value in the sense that they provide key building blocks for some of the conclusions lower down.

(C1): Conditioning to remain negative: *There exists a family of probability measures* $\hat{\mathbf{P}}^{\downarrow} = \{\hat{\mathbf{P}}^{\downarrow}_{y,\theta} : y \leq 0, \theta \in \mathcal{S}\}$ *such that* $((\xi, \Theta), \hat{\mathbf{P}}^{\downarrow})$ *is a right continuous Markov process taking values in* $(-\infty, 0] \times \mathcal{S}$*. Moreover, For all* $y < 0$, $\theta \in \mathcal{S}$, $t \geq 0$ *and* $\Lambda \in \mathcal{F}_t$,

$$\hat{\mathbf{P}}^{\downarrow}_{y,\theta}(\Lambda) = \lim_{q \to 0+} \hat{\mathbf{P}}_{y,\theta}\left(\Lambda, t < e_q \,|\, \tau_0^+ > e_q\right),$$

where (ξ, Θ) *under* $\hat{\mathbf{P}}_{y,\theta}$ *is equal in law to* $(-\xi, \Theta)$*, when* $-\xi_0 = y \in \mathbb{R}$*, and* $\Theta_0 = \theta \in \mathcal{S}$*,* e_q *is an independent and exponentially distributed random variable with parameter* q *and* $\tau_0^+ = \inf\{t > 0 : \xi_t > 0\}$.

(C2): Stationary overshoots and undershoots: *For every* $\theta \in \mathcal{S}$*, the joint probability measures on* $\mathcal{S} \times \mathbb{R}^- \times \mathcal{S} \times \mathbb{R}^+$

$$\mathbf{P}_{0,\theta}\left(\Theta_{\tau_x^+-} \in \mathrm{d}v, \xi_{\tau_x^+-} - x \in \mathrm{d}y, \Theta_{\tau_x^+} \in \mathrm{d}\phi, \xi_{\tau_x^+} - x \in \mathrm{d}z\right)$$

converges weakly to a probability measure $\rho(\mathrm{d}v, \mathrm{d}y, \mathrm{d}\phi, \mathrm{d}z)$ *as* $x \to +\infty$*. In particular,* $\mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+} - x \in \mathrm{d}z,\ \Theta_{\tau_x^+} \in \mathrm{d}\phi\right)$ *converges weakly to a probability measure denoted by* $\rho^{\ominus}(\mathrm{d}z, \mathrm{d}\phi)$ *and* $\mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+-} - x \in \mathrm{d}y,\ \Theta_{\tau_x^+-} \in \mathrm{d}v\right)$ *converges weakly to a probability measure denoted by* $\rho^{\oplus}(\mathrm{d}y, \mathrm{d}v)$.

As alluded to above, we can use the former two main conclusions above to build a process which acts as an entrance law of the ssMp from the origin.

(C3): Candidate entrance law: *Let* $\mathbb{P}^{\searrow}_z$ *denote the law of* X *given by the Lamperti-Kiu transform (1.1) under* $\hat{\mathbf{P}}^{\downarrow}_{y,\theta}$ *with* $y = \log\|z\|$ *and* $\theta = \arg z$*, and let* ϱ *denote the image measure of* $\rho^{\oplus}$ *under the map* $(y, \theta) \mapsto y\mathrm{e}^{\theta}$*. Then the process* $(X, \mathbb{P}^{\searrow}_{\varrho})$ *has a finite lifetime* $\bar{\zeta}$ *with* $X_{\bar{\zeta}-} = 0$*. Its time reversal process* $((\widetilde{X}_t := X_{(\bar{\zeta}-t)-})_{t<\bar{\zeta}}, \mathbb{P}^{\searrow}_{\varrho})$ *is a right-continuous Markov process satisfying that* $\widetilde{X}_0 = 0$ *and* $\widetilde{X}_t \neq 0$ *for all* $t > 0$*. Moreover,* $((\tilde{X}_t)_{0<t<\bar{\zeta}}, \mathbb{P}^{\searrow}_{\varrho})$ *is a strong Markov process having the same transition rates as the ssMp* $(X, \{\mathbb{P}_z, z \in \mathcal{H}\})$ *killed when exiting the unit ball.*

Moreover the stability of the overshoots and undershoots in the second main conclusion also helps with identifying the above candidate entrance law as unique in the sense of weak limits on the Skorokhod space.

(C4): Uniqueness of the entrance law: *There exists a probability measure* $\mathbb{P}_0$ *such that*

1. *w-* $\lim_{z\to 0}\mathbb{P}_z = \mathbb{P}_0$ *in the weak sense of measures on the Skorokhod space.*
2. $(X, \{\mathbb{P}_z, z \in \mathcal{H}_0\})$ *is a ssMp.*
3. $(X, \{\mathbb{P}_z, z \in \mathcal{H}_0\})$ *is a Feller process.*
4. $((X_t)_{t<\tau_r^\ominus}, \mathbb{P}_0)$ *is equal in law with* $(rX_{(\bar{\zeta}-r^{-\alpha}t)-})_{t<r^\alpha\bar{\zeta}}, \mathbb{P}^{\searrow}_{\varrho})$ *for every* $r > 0$.
5. *Under* $\mathbb{P}_0$ *the process* X *starts at* 0 *and leaves* 0 *instantaneously.*

Here $\tau_r^\ominus = \inf\{t > 0 : \ \|X_t\| > r\}$. *Moreover,* $\mathbb{P}_0$ *is the unique probability measure such that the extension* $(X, \{\mathbb{P}_z, z \in \mathcal{H}_0\})$ *is a right continuous Markov process satisfying either (3) or (5) listed above.*

Finally we can reassert the stability of the underlying MAP over/undershoots to generate the unique entrance law at the origin, but now in terms of the ssMp.

(C5): Stability of the the process started at the origin: *For every* $\delta > 0$, $((X_{\tau_\delta^\ominus -}, X_{\tau_\delta^\ominus}), \mathbb{P}_z)$ *converges in distribution to* $((X_{\tau_\delta^\ominus -}, X_{\tau_\delta^\ominus}), \mathbb{P}_0)$ *as* $z \to 0$, *and*

$$\begin{aligned} w\text{-}\lim_{\mathcal{H}\ni z\to 0} \mathbb{P}_z &\Big(\arg(X_{\tau_1^\ominus -}) \in \mathrm{d}v,\ \log\|X_{\tau_1^\ominus -}\| \in \mathrm{d}y,\ \arg(X_{\tau_1^\ominus}) \in \mathrm{d}\phi,\ \log\|X_{\tau_1^\ominus}\| \in \mathrm{d}z\Big) \\ &= \mathbb{P}_0 \Big(\arg(X_{\tau_1^\ominus -}) \in \mathrm{d}v,\ \log\|X_{\tau_1^\ominus -}\| \in \mathrm{d}y,\ \arg(X_{\tau_1^\ominus}) \in \mathrm{d}\phi,\ \log\|X_{\tau_1^\ominus}\| \in \mathrm{d}z\Big) \\ &= \rho(\mathrm{d}v, \mathrm{d}y, \mathrm{d}\phi, \mathrm{d}z). \end{aligned}$$

In the case $d = 1$ and the ssMp is positive, several works have established the limit $\mathbb{P}_0 = \text{w-}\lim_{z\to 0}\mathbb{P}_z$ using various techniques, see [7, 8, 9, 10, 30]. Recently, in the case when ssMp is allowed to take negative values as well, entrance laws were obtained in [16]. Our contribution here is two-fold. Firstly we show, under suitable conditions, the existence of an entrance law at 0 for an ssMp in any dimension. Secondly, our proof here uses a path reversal argument which follows the spirit of [8, 16], but works directly with the reversal of the ssMp rather than the underlying MAP. We note that this approach in dimension $d = 1$ or $d = 1/2$ (i.e. positive self-similar Markov processes), taking all fluctuation theory for granted in those settings (which means fluctuation theory of Lévy processes for $d = 1/2$), our approach offers an alternative simple proof of the entrance laws.

The rest of this paper is structured as follows. In Section 2 we develop the fluctuation theory for general MAPs, which we believe is of independent interest and should be useful in studying ssMps. In Section 3 we present the notions of duality as well as several time-reversal results about duality. Among them, Lemma 3.2 plays a key role in our path reversal argument. In Section 6, we present our working assumptions and the main result, Theorem 6.1, which gives the existence of a weak limit of $\mathbb{P}_z$ as $z \to 0$, as well as the explicit law of the process started at the origin. Our main result is proved step by step through the arguments in Sections 4-8: Firstly we define a family of probability measures $\{\hat{\mathbf{P}}^{\downarrow}_{x,\theta}, x \le 0, \theta \in \mathcal{S}\}$ under which the MAP (ξ, Θ) is conditioned to stay negative. Then we show both the overshoots and undershoots of the MAP (ξ, Θ) have stationary distributions, which we denote by $\rho^\ominus$ and $\rho^\oplus$ respectively. Starting from $((\xi, \Theta), \hat{\mathbf{P}}^{\downarrow}_{\rho^\oplus})$ we construct by Lamperti-Kiu transform the process $(X, \mathbb{P}^{\searrow}_{\varrho})$ which is conditioned to stay inside the unit ball and hit the origin in a finite time. By time-reversing $(X, \mathbb{P}_{\varrho}^{\searrow})$ from its lifetime, we get the law of $(X, \mathbb{P}_0)$ until first exit from a unit ball. Finally we prove $\mathbb{P}_0$ is the weak limit of $\mathbb{P}_z$ as $z \to 0$.

Notations: Throughout this paper, we use ":=" as definition and "$\stackrel{d}{=}$" to mean "equal in distribution". For a Polish space (E, d), $\mathbb{D}_E[0, T)$ denotes the space of functions $\omega : [0, T) \to E \cup \{\partial\}$, where ∂ is a cemetery state, such that there exists $\zeta = \zeta(\omega) \in [0, T]$, called the lifetime of ω, with the property that $t \mapsto \omega(t)$ is a càdlàg function from $[0, \zeta)$ to E and $\omega(t) = \partial$ for $t \geq \zeta$. We endow the space $\mathbb{D}_E[0, T)$ with the Skorokhod topology which makes it into a Polish space. We use the shorthand notation $\mathbb{D}_E = \mathbb{D}_E[0, \infty)$. Every function on E is automatically extended to $E \cup \{\partial\}$ by setting $f(\partial) = 0$. For a point $x \in \mathbb{R}^d$, we use $\|x\|$ to denote its Euclidean norm. For $q > 0$, we use $\mathbf{e}_q$ to denote an independent exponential random variable with mean $1/q$.

Part II
Fluctuation theory of MAPs

2. Preliminaries.

2.1. *Markov additive processes and Lévy systems.* Suppose $(\xi_t, \Theta_t)_{t\geq 0}$ is the coordinate process in $\mathbb{D}_{\mathbb{R}\times\mathcal{S}}$ and $((\xi, \Theta), \mathbf{P}) = ((\xi_t, \Theta_t)_{t\geq 0}, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, \{\mathbf{P}_{x,\theta} : (x, \theta) \in \mathbb{R} \times \mathcal{S}\})$ is a (possibly killed) Markov process with $\mathbf{P}_{x,\theta} (\xi_0 = x, \Theta_0 = \theta) = 1$. Here $(\mathcal{F}_t)_{t\geq 0}$ is the minimal augmented admissible filtration and $\mathcal{F}_\infty = \bigvee_{t=0}^{+\infty} \mathcal{F}_t$.

Definition 2.1. The process $((\xi, \Theta), \mathbf{P})$ is called a Markov additive process (MAP) on $\mathbb{R} \times \mathcal{S}$ if, for any $t \geq 0$, given $\{(\xi_s, \Theta_s), s \leq t\}$, the process $(\xi_{s+t} - \xi_t, \Theta_{s+t})_{s\geq 0}$ has the same law as $(\xi_s, \Theta_s)_{s\geq 0}$ under $\mathbf{P}_{0,v}$ with $v = \Theta_t$. We call $((\xi, \Theta), \mathbf{P})$ a nondecreasing MAP if ξ is a nondecreasing process on $\mathbb{R}$.

For a MAP process $((\xi, \Theta), \mathbf{P})$, we call ξ the *ordinate* and Θ the *modulator*. By definition we can see that a MAP is translation invariant in ξ, i.e., $((\xi_t, \Theta_t)_{t\geq 0}, \mathbf{P}_{x,\theta})$ is equal in law to $((\xi_t + x, \Theta_t)_{t\geq 0}, \mathbf{P}_{0,\theta})$ for all $x \in \mathbb{R}$ and $\theta \in \mathcal{S}$.

We assume throughout the paper that $(\Theta_t)_{t\geq 0}$ is a Hunt process and $(\xi_t)_{t\geq 0}$ is quasi-left continuous on $[0, \zeta)$. Then it is shown in [14] that there exist a continuous increasing additive functional $t \mapsto H_t$ of Θ and a transition kernel Π from $\mathcal{S}$ to $\mathcal{S} \times \mathbb{R}$ satisfying

$$\Pi(\theta, \{(\theta, 0)\}) = 0, \qquad \int_{\mathbb{R}} \left(1 \wedge |y|^2\right) \Pi(\theta, \{\theta\} \times \mathrm{d}y) < +\infty \quad \forall \theta \in \mathcal{S},$$

such that, for every nonnegative measurable function $f : \mathcal{S} \times \mathcal{S} \times \mathbb{R} \to \mathbb{R}^+$, every $\theta \in \mathcal{S}$ and

$t \geq 0$,

$$\mathbf{P}_{0,\theta}\left[\sum_{s\leq t} f(\Theta_{s-},\Theta_s,\xi_s-\xi_{s-})\mathbb{1}_{\{\Theta_{s-}\neq\Theta_s \text{ or } \xi_{s-}\neq\xi_s\}}\right] = \mathbf{P}_{0,\theta}\left[\int_0^t \mathrm{d}H_s \int_{\mathcal{S}\times\mathbb{R}} \Pi(\Theta_s,\mathrm{d}v,\mathrm{d}y) f(\Theta_s,v,y)\right].$$

This pair (H,Π) is said to be a *Lévy system* for $((\xi,\Theta),\mathbf{P})$. It can be shown that for every nonnegative predictable process Z and nonnegative measurable function $g:\mathcal{S}\times\mathbb{R}\times\mathcal{S}\times\mathbb{R}\to\mathbb{R}^+$,

$$\mathbf{P}_{0,\theta}\left[\sum_{s\leq t} Z_s g(\Theta_{s-},\xi_{s-},\Theta_s,\xi_s)\mathbb{1}_{\{\Theta_{s-}\neq\Theta_s \text{ or } \xi_{s-}\neq\xi_s\}}\right] = \mathbf{P}_{0,\theta}\left[\int_0^t \mathrm{d}H_s Z_s \int_{\mathcal{S}\times\mathbb{R}} \Pi(\Theta_s,\mathrm{d}v,\mathrm{d}y) g(\Theta_s,\xi_s,v,\xi_s+y)\right] \tag{2.1}$$

for all $\theta\in\mathcal{S}$ and $t\geq 0$.

The topic of MAPs are covered in various parts of the literature. We refer to [15, 14, 4, 5, 12, 25] to name but a few of the texts and papers which give a general treatment.

For the remainder of the paper we will restrict ourselves to the setting that, up to killing of the MAP, $H_t = t$. On account of the bijection in (1.2), this naturally puts us in a restricted class of self-similar Markov processes through the underlying driving MAP, however, as we will shortly see, it is on the MAP that we will impose additional assumptions.

2.2. *Fluctuation theory for MAPs.*

Definition 2.2. For any $y\in\mathbb{R}$, let $\tau_y^+ := \inf\{t>0:\xi_t>y\}$. We say that $((\xi,\Theta),\mathbf{P})$ is upwards regular if

$$\mathbf{P}_{0,\theta}\left(\tau_0^+ = 0\right) = 1 \quad \forall\theta\in\mathcal{S}.$$

Suppose $(X,\mathbb{P}) = ((X_t)_{t\geq 0},\{\mathbb{P}_z : z\in\mathcal{H}\})$ is the ssMp associated to the MAP $((\xi,\Theta),\mathbf{P})$ via Lamperti-Kiu transform. We say $(X,\mathbb{P})$ is *sphere-exterior regular* if $((\xi,\Theta),\mathbf{P})$ is upwards regular. For $r>0$ let $\tau_r^{\ominus} := \inf\{t>0: \ \|X_t\|>r\}$. Immediately by the definition, $(X,\mathbb{P})$ is sphere-exterior regular if and only if $\mathbb{P}_z\left(\tau_1^{\ominus}=0\right)=1$ for all $z\in\mathcal{H}$ with $\|z\|=1$.

In the remaining of this paper we assume that the MAP $((\xi,\Theta),\mathbf{P})$ is upwards regular. This assumption is not really necessary but nevertheless avoids a lot of unnecessary technicalities when we explore the fluctuation properties.

2.2.1. *Excursion from maximum/minimum.* Let $\bar{\xi}_t := \sup_{s\leq t}\xi_s$ and $U_t := \bar{\xi}_t - \xi_t$. Then under $\mathbf{P}_{0,\theta}$ the process $(\Theta_t,\xi_t,U_t)_{t\geq 0}$ is an $\mathcal{S}\times\mathbb{R}\times\mathbb{R}^+$-valued right process started at $(\theta,0,0)$, whose transition semigroup on $(0,+\infty)$ is given by

$$P_t f(v,x,u) := \mathbf{P}_{0,v}\left[f\left(\Theta_t,\xi_t+x,u\vee\bar{\xi}_t-\xi_t\right)\right]$$

for every $t \geq 0$ and every nonnegative measurable function $f : \mathcal{S} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$. We shall work with the canonical realization of $(\Theta_t, \xi_t, U_t)_{t\geq 0}$ on the sample space $\mathbb{D}_{\mathcal{S}\times\mathbb{R}\times\mathbb{R}}$.

We define $\bar{M} := \{t \geq 0 : U_t = 0\}$ and $\bar{M}^{cl}$ its closure in $\mathbb{R}^+$. Obviously the set $\mathbb{R}^+ \setminus \bar{M}^{cl}$ is an open set and can be written as a union of intervals. We use $\bar{G}$ and $\bar{D}$, respectively, to denote the sets of left and right end points of such intervals. Define $\bar{R} := \inf\{t > 0 :\ t \in \bar{M}^{cl}\}$. The upwards regularity implies that every point in $\mathcal{S}$ is regular for $\bar{M}$ in the sense of [23]. Thus by [23, Theorem (3.10)] there exist a continuous additive functional $t \mapsto \bar{L}_t$ of $(\Theta_t, U_t)_{t\geq 0}$ which is carried by $\mathcal{S} \times \mathbb{R} \times \{0\}$ and a kernel $\mathfrak{P}$ from $\mathcal{S} \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{D}_{\mathcal{S}\times\mathbb{R}\times\mathbb{R}}$ satisfying $\mathfrak{P}^{\theta,x,u}\left((\Theta_0, \xi_0, U_0) \neq (\theta, x, u)\right) = 0$, $\mathfrak{P}^{\theta,x,u}\left(\bar{R} = 0\right) = 0$ and $\mathfrak{P}^{\theta,x,u}\left(1 - \mathrm{e}^{-\bar{R}}\right) \leq 1$ such that

$$\mathbf{P}_{0,\theta}\left[\sum_{s\in\bar{G}} Z_s f \circ \theta_s\right] = \mathbf{P}_{0,\theta}\left[\int_0^{+\infty} Z_s \mathfrak{P}^{\Theta_s,\xi_s,0}(f)\mathrm{d}\bar{L}_s\right] \tag{2.2}$$

for any predictable process Z and any nonnegative measurable function f on $\mathcal{S} \times \mathbb{R} \times \mathbb{R}$. Moreover, under $\mathfrak{P}^{\theta,x,u}$, the process $(\Theta_t, \xi_t, U_t)_{t>0}$ is a strong Markov process with semigroup $(P_t)_{t\geq 0}$ defined above. In particular, if f is measurable with respect to $\sigma((\Theta_t, U_t)_{t\geq 0})$, then the right-hand side of (2.2) equals

$$\mathbf{P}_{0,\theta}\left[\int_0^{+\infty} Z_s \mathfrak{P}^{\Theta_s,0}(f)\mathrm{d}\bar{L}_s\right]$$

where $\mathfrak{P}^{\theta,u}$ denotes the transition kernel defined for the process $(\Theta_t, U_t)_{t\geq 0}$. It is known (see, for example, [24, Section 3]) that there is a nonnegative measurable function $\ell^+ : \mathcal{S} \to \mathbb{R}^+$ such that

$$\int_0^t 1_{\{s\in\bar{M}\}}\mathrm{d}s = \int_0^t 1_{\{s\in\bar{M}^{cl}\}}\mathrm{d}s = \int_0^t \ell^+(\Theta_s)\mathrm{d}\bar{L}_s \quad \forall t \geq 0 \quad \mathbf{P}_{0,\theta}\text{-a.s.} \tag{2.3}$$

Let $\bar{L}_t^{-1}$ be the right inverse process of $\bar{L}_t$. Define $\xi_t^+ := \xi_{\bar{L}_t^{-1}}$ and $\Theta_t^+ := \Theta_{\bar{L}_t^{-1}}$ for all t such that $\bar{L}_t^{-1} < +\infty$ and otherwise ξ_t^+ and Θ_t^+ are both assigned to be the cemetery state ∂. One can easily verify that $(\bar{L}_t^{-1}, \xi_t^+, \Theta_t^+)_{t\geq 0}$, $(\xi_t^+, \Theta_t^+)_{t\geq 0}$ and $(\bar{L}_t^{-1}, \Theta_t^+)_{t\geq 0}$ are all MAPs where Θ^+ is the modulator. These three processes are referred to as *ascending ladder process*, *ascending ladder height processes* and *ascending ladder time process*, respectively.

Suppose the set $\mathbb{R}^+ \setminus \bar{M}^{cl}$ is written as a union of random intervals (g, d). For such intervals, define

$$(\epsilon_s^{(g)}, \nu_s^{(g)}) := \begin{cases} (U_{g+s}, \Theta_{g+s}) & \text{if } 0 \leq s < d - g, \\ (U_d, \Theta_d) & \text{if } s \geq d - g. \end{cases}$$

$(\epsilon_s^{(g)}, \nu_s^{(g)})_{s\geq 0}$ is called an excursion from the maximum and $\zeta^{(g)} := d - g$ is called its lifetime. We use $\mathcal{E}$ to denote the collection $\{(\epsilon_s^{(g)}(\omega), \nu_s^{(g)}(\omega))_{s\geq 0} : g \in \bar{G}(\omega),\ \omega \in \mathbb{D}_{\mathcal{S}\times\mathbb{R}\times\mathbb{R}}\}$, and call it the space of excursions. Let n_θ^+ be the image measure of $\mathfrak{P}^{\theta,0}$ under the mapping that stops the path of $(\Theta_t, U_t)_{t\geq 0}$ at time $\bar{R}$. A direct consequence of [26, equation (4.9)] is that for any bounded measurable functionals $F : \mathbb{D}_{\mathbb{R}\times\mathcal{S}} \to \mathbb{R}$ and $G : \mathbb{R}^+ \times \mathbb{D}_{\mathbb{R}\times\mathcal{S}} \to \mathbb{R}$,

$$\mathbf{P}_{y,\theta}\left[\sum_{g\in\bar{G}} G\big(g,(\xi_t,\Theta_t)_{t\le g}\big)F\big(\epsilon^{(g)},\vartheta^{(g)}\big)\right]$$
$$= \mathbf{P}_{y,\theta}\left[\int_0^\infty \mathrm{d}\bar{L}_s\, G\big(s,(\xi_t,\Theta_t)_{t\le s}\big)\int_{\mathcal{E}} \mathrm{n}^+_{\Theta_s}(\mathrm{d}\epsilon,\mathrm{d}\vartheta)F(\epsilon,\vartheta)\right]. \tag{2.4}$$

We call $\{\mathrm{n}^+_\theta : \theta\in\mathcal{S}\}$ the *excursion measures at the maximum.*

The excursion measures at the minimum, descending ladder process are defined analogously replacing ξ by $-\xi$.

2.2.2. *Fluctuation identities.* For $t > 0$, define

$$\bar{g}_t := \sup\{s\le t : s\in \bar{M}^{cl}\} \quad \text{and} \quad \bar{\Theta}_t := \Theta_{\bar{g}_t}\mathbb{1}_{\{\bar{\xi}_t=\xi_{\bar{g}_t}\}} + \Theta_{\bar{g}_t-}\mathbb{1}_{\{\bar{\xi}_t>\xi_{\bar{g}_t}\}}.$$

By the right continuity of sample paths one can easily show that $\bar{g}_t$ is equal to $\sup\{s\le t : s\in\bar{M}\}$ with probability 1. Since by quasi-left continuity, $\mathbf{P}_{0,\theta}(\xi_t\neq\xi_{t-}) = 0$ for all $t>0$, we also have $\mathbf{P}_{0,\theta}\left(\bar{g}_t = \sup\{s<t : s\in\bar{M}\}\right) = 1$ for all $t>0$. We claim that $\mathbf{P}_{0,\theta}$-almost surely $\bar{g}_t$ is not a jump time of the process (ξ,Θ) for every $\theta\in\mathcal{S}$. Otherwise one can construct a stopping time T such that

$$\mathbf{P}_{0,\theta}\left(T\in\bar{G}\cap\bar{M}^{cl} :\ \xi_{T-}\neq\xi_T \text{ or } \Theta_{T-}\neq\Theta_T\right) > 0.$$

Note that (2.2) implies $\mathbf{P}_{0,\theta}\left(\sum_{g\in\bar{G}}\mathbb{1}_{\{U_g>0 \text{ or } \Theta_{g-}\neq\Theta_g\}}\right) = 0$. We get from the above inequality that $\mathbf{P}_{0,\theta}\left(T\in\bar{G}\cap\bar{M}^{cl} :\ \xi_{T-}<\xi_T\right) > 0$. This brings a contradiction, since if we apply Markov property and upwards regularity at T, we get $\xi_{T+s} > \xi_T > \xi_{T-}$ for s sufficiently small on the event $\{T\in\bar{G}\cap\bar{M}^{cl},\ \xi_{T-}<\xi_T\}$, which is impossible.

The following identity is one of the key tools in extending identities from the fluctuation theory for Lévy processes to MAPs, it is the base to establish a Wiener-Hopf type factorisation for MAPs.

Proposition 2.3. *Suppose that $((\xi,\Theta),\mathbf{P})$ is a Markov additive process taking values in $\mathbb{R}\times\mathcal{S}$. Then for every bounded measurable functions $F,G : [0,\infty)\times\mathbb{R}\times\mathcal{S}\to\mathbb{R}$ and every $\theta\in\mathcal{S}$,*

$$\mathbf{P}_{0,\theta}\left[G\big(\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q},\bar{\Theta}_{e_q}\big)F\big(\mathbf{e}_q-\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q},\Theta_{\mathbf{e}_q}\big)\right]$$
$$= \int_{\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^+} \mathrm{e}^{-qr}G(r,z,v)\left[q\ell^+(v)F(0,0,v) + \mathrm{n}^+_v\left(F(e_q,\epsilon_{e_q},\nu_{e_q})\mathbb{1}_{\{e_q<\zeta\}}\right)\right]V^+_\theta(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z),$$

where

$$V^+_\theta(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z) := \mathbf{P}_{0,\theta}\left[\int_0^{\bar{L}_\infty}\mathbb{1}_{\{\bar{L}^{-1}_s\in\mathrm{d}r,\ \Theta^+_s\in\mathrm{d}v,\ \xi^+_s\in\mathrm{d}z\}}\mathrm{d}s\right].$$

Proof. It is known from the above argument that $\mathbf{P}_{0,\theta}$-almost surely $\bar{g}_{e_q}$ is not jumping time of (ξ,Θ), and thus $(\bar{\xi}_{e_q},\bar{\Theta}_{e_q})=(\xi_{\bar{g}_{e_q}},\Theta_{\bar{g}_{e_q}})$ $\mathbf{P}_{0,\theta}$-a.s. Then we have

$$\begin{aligned}
&\mathbf{P}_{0,\theta}\left[F\big(\mathbf{e}_q-\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q},\Theta_{\mathbf{e}_q}\big)G\big(\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q},\bar{\Theta}_{e_q}\big)\right]\\
&\quad=\mathbf{P}_{0,\theta}\left[F\big(\mathbf{e}_q-\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q},\Theta_{\mathbf{e}_q}\big)G\big(\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q},\bar{\Theta}_{e_q}\big)\mathbb{1}_{\{\xi_{\mathbf{e}_q}=\bar{\xi}_{\mathbf{e}_q}\}}\right]\\
&\qquad+\mathbf{P}_{0,\theta}\left[F\big(\mathbf{e}_q-\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q},\Theta_{\mathbf{e}_q}\big)G\big(\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q},\bar{\Theta}_{e_q}\big)\mathbb{1}_{\{\xi_{\mathbf{e}_q}<\bar{\xi}_{\mathbf{e}_q}\}}\right]\\
&\quad=\mathbf{P}_{0,\theta}\left[F\big(0,0,\Theta_{\mathbf{e}_q}\big)G\big(\mathbf{e}_q,\xi_{\mathbf{e}_q},\Theta_{\mathbf{e}_q}\big)\mathbb{1}_{\{\xi_{\mathbf{e}_q}=\bar{\xi}_{\mathbf{e}_q}\}}\right]\\
&\qquad+\mathbf{P}_{0,\theta}\left[\sum_{g\in\bar{G}}\mathbb{1}_{\{g<\mathbf{e}_q<g+\zeta^{(g)}\}}F\big(\mathbf{e}_q-g,\epsilon^{(g)}_{\mathbf{e}_q-g},\vartheta^{(g)}_{\mathbf{e}_q-g}\big)G\big(g,\bar{\xi}_g,\Theta_g\big)\right]\\
&\quad=\mathbf{P}_{0,\theta}\left[F\big(0,0,\Theta_{\mathbf{e}_q}\big)G\big(\mathbf{e}_q,\xi_{\mathbf{e}_q},\Theta_{\mathbf{e}_q}\big)\mathbb{1}_{\{\xi_{\mathbf{e}_q}=\bar{\xi}_{\mathbf{e}_q}\}}\right]\\
&\qquad+\mathbf{P}_{0,\theta}\left[\int_0^\infty \mathrm{d}\bar{L}_s\mathbb{1}_{\{s<\mathbf{e}_q\}}G\big(s,\bar{\xi}_s,\Theta_s\big)\mathrm{n}^+_{\Theta_s}\left(F\big(\mathbf{e}_q-s,\epsilon_{\mathbf{e}_q-s},\vartheta_{\mathbf{e}_q-s}\big)\mathbb{1}_{\{e_q-s<\zeta\}}\right)\right],
\end{aligned}\tag{2.5}$$

where in the third equality we used the identity (2.4). For the second term we can use the memorylessness of the exponential distribution and a change of variable to yield

$$\begin{aligned}
&\mathbf{P}_{0,\theta}\left[\int_0^\infty \mathrm{d}\bar{L}_s\mathbb{1}_{\{s<\mathbf{e}_q\}}G\big(s,\bar{\xi}_s,\Theta_s\big)\mathrm{n}^+_{\Theta_s}\left(F\big(\mathbf{e}_q-s,\epsilon_{\mathbf{e}_q-s},\vartheta_{\mathbf{e}_q-s}\big)\mathbb{1}_{\{e_q-s<\zeta\}}\right)\right]\\
&\quad=\mathbf{P}_{0,\theta}\left[\int_0^\infty \mathrm{d}\bar{L}_s\mathrm{e}^{-qs}G\big(s,\bar{\xi}_s,\Theta_s\big)\mathrm{n}^+_{\Theta_s}\left(F\big(\mathbf{e}_q,\epsilon_{\mathbf{e}_q},\vartheta_{\mathbf{e}_q}\big)\mathbb{1}_{\{e_q<\zeta\}}\right)\right]\\
&\quad=\mathbf{P}_{0,\theta}\left[\int_0^{\bar{L}_\infty} \mathrm{d}s\mathrm{e}^{-q\bar{L}_s^{-1}}G\big(\bar{L}_s^{-1},\xi^+_s,\Theta^+_s\big)\mathrm{n}^+_{\Theta^+_s}\left(F\big(\mathbf{e}_q,\epsilon_{e_q},\vartheta_{e_q}\big)\mathbb{1}_{\{e_q<\zeta\}}\right)\right].
\end{aligned}\tag{2.6}$$

For the first term in (2.5) we use (2.3) to get

$$\begin{aligned}
&\mathbf{P}_{0,\theta}\left[F\big(0,0,\Theta_{\mathbf{e}_q}\big)G\big(\mathbf{e}_q,\xi_{\mathbf{e}_q},\Theta_{\mathbf{e}_q}\big)\mathbb{1}_{\{\xi_{\mathbf{e}_q}=\bar{\xi}_{\mathbf{e}_q}\}}\right]\\
&\quad=q\mathbf{P}_{0,\theta}\left[\int_0^{+\infty}\mathrm{e}^{-qt}G(t,\bar{\xi}_t,\Theta_t)F(0,0,\Theta_t)\mathbb{1}_{\{t\in\bar{M}\}}\mathrm{d}t\right]\\
&\quad=q\mathbf{P}_{0,\theta}\left[\int_0^{+\infty}\mathrm{e}^{-qt}G(t,\bar{\xi}_t,\Theta_t)F(0,0,\Theta_t)\ell^+(\Theta_t)\mathrm{d}\bar{L}_t\right]\\
&\quad=q\mathbf{P}_{0,\theta}\left[\int_0^{\bar{L}_\infty}\mathrm{e}^{-q\bar{L}_s^{-1}}G(\bar{L}_s^{-1},\xi^+_s,\Theta^+_s)F(0,0,\Theta^+_s)\ell^+(\Theta^+_s)\mathrm{d}s\right].
\end{aligned}\tag{2.7}$$

By plugging (2.6) and (2.7) into (2.5) we get that

$$\begin{aligned}
&\mathbf{P}_{0,\theta}\left[F\big(\mathbf{e}_q-\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q},\Theta_{\mathbf{e}_q}\big)G\big(\bar{g}_{\mathbf{e}_q},\bar{\xi}_{\mathbf{e}_q},\bar{\Theta}_{e_q}\big)\right]\\
&=\mathbf{P}_{0,\theta}\Big[\int_0^{\bar{L}_\infty}\mathrm{d}s\mathrm{e}^{-q\bar{L}_s^{-1}}G\left(\bar{L}_s^{-1},\xi^+_s,\Theta^+_s\right)\left(q\ell^+(\Theta^+_s)F(0,0,\Theta^+_s)+\mathrm{n}^+_{\Theta^+_s}\left(F(e_q,\epsilon_{e_q},\nu_{e_q})\mathbb{1}_{\{e_q<\zeta\}}\right)\right)\Big].
\end{aligned}$$

We have thus proved this proposition. □

COROLLARY 2.4. *For every $\theta \in \mathcal{S}$, we have*

$$\begin{aligned}
&\mathbf{P}_{0,\theta}\left(\bar{\xi}_{e_q} \in \mathrm{d}z,\ \bar{\xi}_{e_q} - \xi_{e_q} \in \mathrm{d}w,\ \Theta_{e_q} \in \mathrm{d}v\right) \\
&= \delta_0(\mathrm{d}w)\ell^+(v)\int_0^{+\infty} q\mathrm{e}^{-qr} V_\theta^+(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z) \\
&+ \int_{(r,u)\in\mathbb{R}^+\times\mathcal{S}} \mathrm{e}^{-qr}\mathrm{n}_u^+\left(\epsilon_{e_q} \in \mathrm{d}w,\ \nu_{e_q} \in \mathrm{d}v,\ e_q < \zeta\right) V_\theta^+(\mathrm{d}r,\mathrm{d}u,\mathrm{d}z).
\end{aligned}$$

The excursion measures allow us to gain some additional insight into the analytical form of the jumping measures of the ascending ladder processes.

PROPOSITION 2.5. *Suppose $((\xi,\Theta),\mathbf{P})$ is a MAP with Lévy system (H,Π) where $H_t = t\wedge\zeta$. Then the ascending ladder process $((\bar{L}^{-1},\xi^+,\Theta^+),\mathbf{P})$ has a Lévy system (H^+,Γ^+) where $H_t^+ = t\wedge\zeta^+$ and*

$$\Gamma^+(\theta,\mathrm{d}v,\mathrm{d}r,\mathrm{d}y) = \delta_0(\mathrm{d}r)\ell^+(\theta)\Pi(\theta,\mathrm{d}v,\mathrm{d}y) + \mathrm{n}_\theta^+\left(\Pi\left(\nu_r,\mathrm{d}v,\epsilon_r+\mathrm{d}y\right), r<\zeta\right)\mathrm{d}r$$

for $\theta, v \in \mathcal{S}$, $r \geq 0$ and $y > 0$. Here ζ^+ denotes the lifetime of (ξ^+,Θ^+). In particular, the ascending ladder height process $((\xi^+,\Theta^+),\mathbf{P})$ has a Lévy system (H^+,Π^+) where $\Pi^+(\theta,\mathrm{d}v,\mathrm{d}y) = \Gamma^+(\theta,\mathrm{d}v,[0,+\infty),\mathrm{d}y)$ for $\theta, v \in \mathcal{S}$ and $y > 0$.

PROOF. To prove this proposition we apply the theory for Lévy systems and time-changed processes developed in [18]. We consider the strong Markov process $Y_t := (\Theta_t,\xi_t,U_t,t)$ on the state space $\mathcal{S}\times\mathbb{R}\times\mathbb{R}\times[0,\infty)$ where $U_t = \bar{\xi}_t - \xi_t$. Recall that $\bar{M} = \{t \geq 0 :\ U_t = 0\}$ and $\bar{R} = \inf\{t > 0 :\ t \in \bar{M}^{cl}\}$. It is known that the local time at the maximum $\bar{L}_t$ is a continuous additive functional carried by $\tilde{F} := \mathcal{S}\times\mathbb{R}\times\{0\}\times[0,\infty)$. The argument in the beginning of this subsection implies that almost surely the "irregular part" (in the sense of [18]) $G^i := \{s \in \bar{G} :\ U_s \neq 0\}$ is an empty set. Let $\tilde{Y}_t := (\Theta_t^+,\xi_t^+,U_t^+,\bar{L}_t^{-1})$ be the time-changed process of Y_t by the inverse local time $\bar{L}_t^{-1}$. It is a right process on the state space $\tilde{F}$, and following the arguments and calculations in [18, Section 5], one can get a Lévy system for this time-changed process. In fact, applying [18, Theorem 5.2] here, we have

$$\begin{aligned}
&\mathbf{P}_{0,\theta}\left[\sum_{s>0} F\left(\Theta_{s-}^+,\xi_{s-}^+,\bar{L}_{s-}^{-1},\Theta_s^+,\xi_s^+,\bar{L}_s^{-1}\right)\mathbb{1}_{\{\xi_{s-}^+\neq\xi_s^+\}}\right] \\
&= \mathbf{P}_{0,\theta}\left[\int_0^{+\infty}\mathrm{d}s\int_{\mathcal{S}\times\mathbb{R}\times[0,\infty)} F(\Theta_s^+,\xi_s^+,s,v,y,u)\mathbb{1}_{\{\xi_s^+\neq y\}}\Big(\mathfrak{P}^{\Theta_s^+,\xi_s^+,0,s}\left(\Theta_{\bar{R}}\in\mathrm{d}v,\ \xi_{\bar{R}}\in\mathrm{d}y,\bar{R}\in\mathrm{d}u\right)\right. \\
&\qquad\qquad \left. +\ \ell^+(\Theta_s^+)\Pi(\Theta_s^+,\mathrm{d}v,\mathrm{d}y-\xi_s^+)\delta_s(\mathrm{d}u)\Big)\right].
\end{aligned} \tag{2.8}$$

for every nonnegative measurable function F. Here $\mathfrak{P}^{\theta,x,0,s}$ denotes the kernel $\mathfrak{P}^{\theta,x,0}$ trivially extended to include the pure drift process issued from s. So, note that under $\mathfrak{P}^{\theta,x,0,s}$ the process $(Y_t)_{t>0}$ is a Markov process with the same transition rates as $(Y,\mathbf{P}_{x,\theta,s})$. Using this and the translation invariance, we have

$$\mathfrak{P}^{\theta,x,0,s}\left[\mathbb{1}_{\{s<\bar{R},\xi_{\bar{R}}\neq x\}}f(\Theta_{\bar{R}},\xi_{\bar{R}},\bar{R})\right] = \mathfrak{P}^{\theta,0,0}\left[\mathbb{1}_{\{s<\bar{R}\}}\mathbf{P}_{\xi_s,\Theta_s}\left(f(\Theta_{\tau_0^+},\xi_{\tau_0^+}+x,s+\tau_0^+)\mathbb{1}_{\{\xi_{\tau_0^+}>0\}}\right)\right], \tag{2.9}$$

for any $s>0$ and nonnegative measurable function f. Since (ξ,Θ) has Lévy system (H,Π) with $H_t=t\wedge\zeta$, we have

$$\mathbf{P}_{z,v}\left[f(\Theta_{\tau_0^+},\xi_{\tau_0^+}+x,\tau_0^+)1_{\{\xi_{\tau_0^+}>0\}}\right]=\mathbf{P}_{z,v}\left[\int_0^{\tau_0^+}\mathrm{d}s\int_{(-\xi_s,+\infty)}f(v,\xi_s+x+y,s)\Pi(\Theta_s,\mathrm{d}v,\mathrm{d}y)\right],$$

where we used that, on the event $\{\xi_{\tau_0^+}>0\}$, τ_0^+ it is the first jump time of ξ, that takes ξ into the positive axis, and we apply (2.1). Plugging this in (2.9), and using the Markov property under $\mathfrak{P}^{\theta,0,0}$, we have

$$\begin{aligned}&\mathfrak{P}^{\theta,x,0,t}\left[1_{\{s<\bar{R},\xi_{\bar{R}}\neq x\}}f(\Theta_{\bar{R}},\xi_{\bar{R}},\bar{R})\right]\\&\quad=\mathfrak{P}^{\theta,0,0,t}\left[1_{\{s<\bar{R}\}}\mathbf{P}_{\Theta_s,-U_s}\left(\int_0^{\tau_0^+}\mathrm{d}r\int_{(U_r,+\infty)}f(v,-U_r+x+y,r)\Pi(\Theta_r,\mathrm{d}v,\mathrm{d}y)\right)\right]\\&\quad=\mathrm{n}_\theta^+\left[1_{\{s<\zeta\}}\int_s^\zeta\mathrm{d}r\int_{(\epsilon_r,+\infty)}f(v,-\epsilon_r+x+y,r+t)\Pi(\nu_r,\mathrm{d}v,\mathrm{d}y)\right].\end{aligned}$$

By letting $s\to 0+$, we get from above equation that

$$1_{\{x\neq y\}}\mathfrak{P}^{\theta,x,0,t}\left(\Theta_{\bar{R}}\in\mathrm{d}v,\ \xi_{\bar{R}}\in\mathrm{d}y,\bar{R}\in\mathrm{d}u\right)=(\mathrm{d}u-t)\mathrm{n}_\theta^+\left[\int_{(\epsilon_r,+\infty)}1_{\{-\epsilon_r+z+x\in\mathrm{d}y\}}\Pi(\nu_r,\mathrm{d}v,\mathrm{d}z)\right].$$

Plugging this in (2.8) yields that

$$\begin{aligned}&\mathbf{P}_{0,\theta}\left[\sum_{s>0}F\left(\Theta_{s-}^+,\xi_{s-}^+,\bar{L}_{s-}^{-1},\Theta_s^+,\xi_s^+,\bar{L}_s^{-1}\right)1_{\{\xi_{s-}^+\neq\xi_s^+\}}\right]\\&=\mathbf{P}_{0,\theta}\left[\int_0^{+\infty}\mathrm{d}s\int_{\mathcal{S}\times(0,+\infty)\times(0,\infty)}F(\Theta_s^+,\xi_s^+,s,v,\xi_s^++y,s+r)\big[\delta_0(\mathrm{d}r)\ell^+(\Theta_s^+)\Pi(\Theta_s^+,\mathrm{d}v,\mathrm{d}y)\right.\\&\left.+\ \mathrm{n}_{\Theta_s^+}^+\left(\Pi\left(\nu_r,\mathrm{d}v,\epsilon_r+\mathrm{d}y\right),r<\zeta\right)\mathrm{d}r\right],\end{aligned}$$

which in turn yields the assertion of this proposition. □

Remark 2.6. Suppose ξ is a non-killed $\mathbb{R}$-valued Lévy process with triplet (a,σ^2,Π) for which 0 is regular for $(0,+\infty)$. This process can be viewed as the projection of a upwards regular MAP (ξ,Θ) where the modulator Θ is equal to a constant. Therefore all the above results we obtained for MAP can be applied to this Lévy process. We use $\mathbf{P}_0$ (resp. $\hat{\mathbf{P}}_0$) to denote the law of ξ (resp. $-\xi$) started from 0. It is a known fact that its ascending ladder process $(\bar{L}_t^{-1},\xi_t^+)_{t\geq 0}$ is a (killed) bivariate subordinator. Let Π^+ be the Lévy measure of ξ^+. Proposition 2.5 yields that for $y>0$,

$$\Pi^+(y,+\infty)=\ell^+\Pi(y,+\infty)+\mathrm{n}^+\left[\int_0^{+\infty}\Pi(\epsilon_r+y,+\infty)\mathrm{d}r\right],\tag{2.10}$$

where ℓ^+ is the drift coefficient of $\bar{L}_t^{-1}$ and n^+ is the excursion measure at maximum. It follows by Proposition 2.3 that for any nonnegative measurable function $F:\mathbb{R}\to\mathbb{R}^+$

$$\begin{aligned}\mathbf{P}_0\left[F(\bar{\xi}_{e_q}-\xi_{e_q})\right]&=\frac{q\ell^+F(0)+\mathrm{n}^+\left[\int_0^\zeta q\mathrm{e}^{-qs}F(\epsilon_s)\mathrm{d}s\right]}{\Phi(q)},\\ \hat{\mathbf{P}}_0\left[F(\bar{\xi}_{e_q})\right]&=\hat{\Phi}(q)\int_{\mathbb{R}^+\times\mathbb{R}^+}\mathrm{e}^{-qr}F(z)\hat{V}^+(\mathrm{d}r,\mathrm{d}z)\end{aligned}\tag{2.11}$$

where $\hat{V}^+(\mathrm{d}r,\mathrm{d}z) := \hat{\mathbf{P}}_0\left[\int_0^{+\infty} \mathbb{1}_{\{\bar{L}_s^{-1}\in \mathrm{d}r, \xi_s^+\in\mathrm{d}z\}}\mathrm{d}s\right]$, and $\Phi(q)$ (resp. $\hat{\Phi}(q)$) is equal to the Laplace exponent of the (killed) subordinator $(\bar{L}_t^{-1})_{t\geq 0}$ under $\mathbf{P}_0$ (resp. $\hat{\mathbf{P}}_0$). The Wiener-Hopf factorization of Lévy process implies that $\Phi(q)\hat{\Phi}(q) = \kappa q$ for some constant $\kappa > 0$. We may and do assume $\kappa = 1$. By this and (2.11), we get

$$\begin{aligned}
\ell^+ F(0) + \mathrm{n}^+\left[\int_0^\zeta F(\epsilon_s)\mathrm{d}s\right] &= \lim_{q\to 0+}\frac{\mathbf{P}_0\left[F(\bar{\xi}_{e_q}-\xi_{e_q})\right]\Phi(q)}{q}\\
&= \lim_{q\to 0+}\frac{\hat{\mathbf{P}}_0\left[F(\bar{\xi}_{e_q})\right]}{\hat{\Phi}(q)}\\
&= \int_{\mathbb{R}^+} F(z)\hat{U}^+(\mathrm{d}z)
\end{aligned}$$

where $\hat{U}^+(\mathrm{d}z) := \hat{\mathbf{P}}_0\left[\int_0^{+\infty}\mathbb{1}_{\{\xi_t^+\in\mathrm{d}z\}}\mathrm{d}t\right]$. In the second equality we use the fact that $(\bar{\xi}_{e_q}-\xi_{e_q}, \mathbf{P}_0) \overset{d}{=} (\bar{\xi}_{e_q}, \hat{\mathbf{P}}_0)$. Setting $F(\cdot) = \Pi(y+\cdot, +\infty)$ in above equation and plugging it in (2.10) we get

$$\Pi^+(y,+\infty) = \int_{\mathbb{R}^+}\Pi(z+y,+\infty)U^-(\mathrm{d}z)$$

for $y > 0$. This is Vigon's identity for Lévy process.

Define

$$U_\theta^+(\mathrm{d}v,\mathrm{d}z) := \mathbf{P}_{0,\theta}\left[\int_0^{\bar{L}_\infty}\mathbb{1}_{\{\Theta_s^+\in\mathrm{d}v,\ \xi_s^+\in\mathrm{d}z\}}\mathrm{d}s\right].$$

PROPOSITION 2.7. *Suppose (ξ,Θ) is a MAP with Lévy system (H,Π) where $H_t = t\wedge\zeta$. Then for any $x>0$, $\theta\in\mathcal{S}$ and any nonnegative measurable functions $f,g:\mathcal{S}\times\mathbb{R}^+\to\mathbb{R}^+$,*

$$\begin{aligned}
&\mathbf{P}_{0,\theta}\left[f(\Theta_{\tau_x^+-}, x-\xi_{\tau_x^+-})g(\Theta_{\tau_x^+},\xi_{\tau_x^+}-x)\mathbb{1}_{\{\xi_{\tau_x^+}>x\}}\right]\\
&\quad= \int_{\mathcal{S}\times[0,x]}U_\theta^+(\mathrm{d}v,\mathrm{d}z)\Big[\ell^+(v)f(v,x-z)G(v,x-z)\\
&\qquad\qquad +\mathrm{n}_v^+\left(\int_0^\zeta f(\nu_s,x-z+\epsilon_s)G(\nu_s,x-z+\epsilon_s)\mathrm{d}s\right)\Big],
\end{aligned}\tag{2.12}$$

where $G(v,u) := \int_{\mathcal{S}\times(u,+\infty)} g(\phi,y-u)\Pi(v,\mathrm{d}\phi,\mathrm{d}y)$ for $v\in\mathcal{S}$ and $u\in\mathbb{R}$. In particular

$$\begin{aligned}
&\mathbf{P}_{0,\theta}\left[g(\Theta_{\tau_x^+},\xi_{\tau_x^+}-x)\mathbb{1}_{\{\xi_{\tau_x^+}>x\}}\right]\\
&\quad= \int_{\mathcal{S}\times[0,x]}U_\theta^+(\mathrm{d}v,\mathrm{d}z)\int_{\mathcal{S}\times(x-z,+\infty)}g(\phi,z+y-x)\Pi^+(v,\mathrm{d}\phi,\mathrm{d}y).
\end{aligned}\tag{2.13}$$

PROOF. Let $\Delta\xi_s := \xi_s - \xi_{s-}$ for any $s > 0$. By (2.1) we have

$$
\begin{aligned}
&\mathbf{P}_{0,\theta}\left[f(\Theta_{\tau_x^+-}, x-\xi_{\tau_x^+-})g(\Theta_{\tau_x^+}, \xi_{\tau_x^+}-x)\mathbb{1}_{\{\xi_{\tau_x^+}>x\}}\right] \\
&\quad= \mathbf{P}_{0,\theta}\left[\sum_{s\geq 0} f(\Theta_{s-}, x-\xi_{s-})g(\Theta_s, \xi_{s-}+\Delta\xi_s - x)\mathbb{1}_{\{\bar{\xi}_{s-}\leq x, \xi_{s-}+\Delta\xi_s - x>0\}}\right] \\
&\quad= \mathbf{P}_{0,\theta}\left[\int_0^\zeta \mathbb{1}_{\{\bar{\xi}_s\leq x\}} f(\Theta_s, x-\xi_s)\mathrm{d}s \int_{\mathcal{S}\times\mathbb{R}^+} g(v, \xi_s + y - x)\mathbb{1}_{\{\xi_s+y-x>0\}}\Pi(\Theta_s, \mathrm{d}v, \mathrm{d}y)\right]
\end{aligned}
$$

$$
= \mathbf{P}_{0,\theta}\left[\int_0^\zeta \mathbb{1}_{\{\bar{\xi}_s\leq x\}} f(\Theta_s, x-\xi_s)G(\Theta_s, x-\xi_s)\mathrm{d}s\right]. \tag{2.14}
$$

We set $F(y,v) := f(v, x-y)G(v, x-y)$, then the right-hand side of (2.14) equals

$$
\begin{aligned}
&\mathbf{P}_{0,\theta}\left[\int_0^\zeta \mathbb{1}_{\{\bar{\xi}_s\leq x\}} F(\xi_s, \Theta_s)\mathrm{d}s\right] \\
&\quad= \mathbf{P}_{0,\theta}\left[\int_0^\zeta \mathbb{1}_{\{\bar{\xi}_s\leq x, s\in\bar{M}^{cl}\}} F(\xi_s, \Theta_s)\mathrm{d}s\right] + \mathbf{P}_{0,\theta}\left[\int_0^\zeta \mathbb{1}_{\{\bar{\xi}_s\leq x,\ s\notin\bar{M}^{cl}\}} F(\xi_s, \Theta_s)\mathrm{d}s\right] \\
&\quad= \mathbf{P}_{0,\theta}\left[\int_0^{+\infty} \mathbb{1}_{\{\bar{\xi}_s\leq x\}} F(\xi_s, \Theta_s)\ell^+(\Theta_s)\mathrm{d}\bar{L}_s\right] + \mathbf{P}_{0,\theta}\left[\sum_{g\in\bar{G}} \mathbb{1}_{\{\bar{\xi}_g\leq x\}}\int_g^d F(\xi_s, \Theta_s)\mathrm{d}s\right].
\end{aligned}
$$

By (2.4) the second term equals

$$
\mathbf{P}_{0,\theta}\left[\int_0^{+\infty} \mathbb{1}_{\{\bar{\xi}_s\leq x\}} \mathrm{n}^+_{\Theta_s}\left(\int_0^\zeta F\left(\bar{\xi}_s - \epsilon_r, \nu_r\right)\right)\mathrm{d}\bar{L}_s\right].
$$

Hence we have

$$
\begin{aligned}
&\mathbf{P}_{0,\theta}\left[f(\Theta_{\tau_x^+-}, x-\xi_{\tau_x^+-})g(\Theta_{\tau_x^+}, \xi_{\tau_x^+}-x)\mathbb{1}_{\{\xi_{\tau_x^+}>x\}}\right] \\
&\quad= \mathbf{P}_{0,\theta}\left[\int_0^{+\infty} \mathbb{1}_{\{\bar{\xi}_s\leq x\}}\left(\ell^+(\Theta_s)F(\xi_s, \Theta_s) + \mathrm{n}^+_{\Theta_s}\left(\int_0^\zeta F\left(\bar{\xi}_s - \epsilon_r, \nu_r\right)\right)\right)\mathrm{d}\bar{L}_s\right] \\
&\quad= \mathbf{P}_{0,\theta}\left[\int_0^{\bar{L}_\infty} \mathbb{1}_{\{\xi^+_s\leq x\}}\left(\ell^+(\Theta^+_s)F(\xi^+_s, \Theta^+_s) + \mathrm{n}^+_{\Theta^+_s}\left(\int_0^\zeta F\left(\xi^+_s - \epsilon_r, \nu_r\right)\right)\right)\mathrm{d}s\right] \\
&\quad= \int_{\mathcal{S}\times[0,x]} U^+_\theta(\mathrm{d}v, \mathrm{d}z)\left(\ell^+(v)F(v,z) + \mathrm{n}^+_v\left(\int_0^\zeta F(z-\epsilon_r, \nu_r)\mathrm{d}r\right)\right),
\end{aligned}
$$

which yields (2.12). (2.13) follows directly from (2.12) and Proposition 2.5. □

We say a path of ξ creeps across level x if it enters $(x, +\infty)$ continuously, that is, the first passage time in $(x, +\infty)$ is not a jumping time. The next lemma we present is about what happens on the event of creeping. It follows from [15, Proposition (1.5) and Theorem (1.7)].

LEMMA 2.1. *Suppose the ascending ladder height process* $((\xi^+, \Theta^+), \mathbf{P})$ *has a Lévy system* (H^+, Π^+) *where* $H^+_t = t\wedge\zeta^+$. *If the continuous part of* ξ^+ *can be represented by*

$\int_0^{t\wedge\zeta^+} a^+(\Theta^+_s)\mathrm{d}s$ *for some nonnegative measurable function* a^+ *on* $\mathcal{S}$, *then for every* $\theta\in\mathcal{S}$, $1\!\!1_{\{a^+(v)>0\}}U^+_\theta(\mathrm{d}v,\mathrm{d}x)$ *has a density* $u^+_\theta(\mathrm{d}v,x)$ *with respect to the Lebesgue measure* $\mathrm{d}x$. *Moreover, if we define* $T^+_x:=\inf\{t>0:\xi^+_t>x\}$, *then for any nonnegative measurable function* $f:\mathcal{S}\times\mathcal{S}\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ *and almost every* $x>0$,

$$\mathbf{P}_{0,\theta}\left(\xi^+_{T^+_x-}<x=\xi^+_{T^+_x}\right)=0, \tag{2.15}$$

and

$$\mathbf{P}_{0,\theta}\left[f\left(\Theta^+_{T^+_x-},\Theta^+_{T^+_x},x-\xi^+_{T^+_x-},\xi^+_{T^+_x}-x\right)1\!\!1_{\{\xi^+_{T^+_x}=x\}}\right]=\int_{\mathcal{S}}a^+(v)f(v,v,0,0)u^+_\theta(\mathrm{d}v,x). \tag{2.16}$$

LEMMA 2.2. *Suppose the MAP* $((\xi,\Theta),\mathbf{P})$ *has a Lévy system* (H,Π) *where* $H_t=t\wedge\zeta$. *If* $(x,\theta)\in(0,+\infty)\times\mathcal{S}$ *satisfies that*

$$\mathbf{P}_{0,\theta}\left(\xi_{\tau^+_x-}<x=\xi_{\tau^+_x}\right)=0, \tag{2.17}$$

then

$$\mathbf{P}_{0,\theta}\left(\Theta_{\tau^+_x-}\neq\Theta_{\tau^+_x},\xi_{\tau^+_x}=x\right)=0.$$

PROOF. For $x>0$, let $\tau_{[x,+\infty)}$ denote the first time when ξ enters $[x,+\infty)$. The upwards regularity of $((\xi,\Theta),\mathbf{P})$ implies that $\tau_{[x,+\infty)}=\tau^+_x$ $\mathbf{P}_{0,\theta}$-a.s. It follows by (2.17) and (2.1) that

$$\begin{aligned}
\mathbf{P}_{0,\theta}\left(\Theta_{\tau^+_x-}\neq\Theta_{\tau^+_x},\ \xi_{\tau^+_x}=x\right) &= \mathbf{P}_{0,\theta}\left(\Theta_{\tau^+_x-}\neq\Theta_{\tau^+_x},\ \xi_{\tau^+_x-}=\xi_{\tau^+_x}=x\right)\\
&= \mathbf{P}_{0,\theta}\left[\sum_{s\geq 0}1\!\!1_{\{\xi_r<x,\forall r\in[0,s),\ \Theta_{s-}\neq\Theta_s,\ \xi_{s-}=\xi_s=x\}}\right]\\
&= \mathbf{P}_{0,\theta}\left[\int_0^{+\infty}1\!\!1_{\{\xi_r<x,\forall r\in[0,s),\ \xi_s=x\}}\Pi(\Theta_s,\mathcal{S}\setminus\{\Theta_s\},\{0\})\mathrm{d}s\right]\\
&= \mathbf{P}_{0,\theta}\left[\int_0^{+\infty}1\!\!1_{\{\tau_{[x,+\infty)}=s,\ \xi_s=x\}}\Pi(\Theta_s,\mathcal{S}\setminus\{\Theta_s\},\{0\})\mathrm{d}s\right]\\
&= 0.
\end{aligned}$$

The last equality is because the integral inside $\mathbf{P}_{0,\theta}$ equals 0. □

PROPOSITION 2.8. *Suppose the MAP* $((\xi,\Theta),\mathbf{P})$ *has a Lévy system* (H,Π) *where* $H_t=t\wedge\zeta$ *and the continuous part of* ξ^+ *can be represented by* $\int_0^{t\wedge\zeta^+}a^+(\Theta^+_s)\mathrm{d}s$ *for some nonnegative measurable function* a^+ *on* $\mathcal{S}$. *Then for every* $\theta\in\mathcal{S}$, *every nonnegative measurable function* $f:\mathcal{S}\times\mathcal{S}\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ *and almost every* $x>0$,

$$\mathbf{P}_{0,\theta}\left[f(\Theta_{\tau^+_x-},\Theta_{\tau^+_x},x-\xi_{\tau^+_x-},\xi_{\tau^+_x}-x)1\!\!1_{\{\xi_{\tau^+_x}=x\}}\right]=\int_{\mathcal{S}}a^+(v)f(v,v,0,0)u^+_\theta(\mathrm{d}v,x), \tag{2.18}$$

where $u^+_\theta(\mathrm{d}v,x)$ *is the density function given in Lemma 2.1.*

Proof. It is easy to see from Proposition 2.5 that the conditions of Lemma 2.1 holds under the assumptions of this proposition. Fix an arbitrary $\theta \in \mathcal{S}$. Let $\mathcal{R}$ denote the set of points for which both identities in Lemma 2.1 hold. Then $\mathrm{Leb}(\mathbb{R}^+ \setminus \mathcal{R})=0$. We note that $(\xi_{\tau_x^+}, \Theta_{\tau_x^+}) = (\xi^+_{T_x^+}, \Theta^+_{T_x^+})$. If we can prove $\mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+-} < x = \xi_{\tau_x^+}\right) = 0$ for every $x \in \mathcal{R}$, then by Lemma 2.2 $\Theta_{\tau_x^+-} = \Theta_{\tau_x^+}$ $\mathbf{P}_{0,\theta}$-a.s. on $\{\xi_{\tau_x^+} = x\}$, and (2.18) is a direct consequence of (2.16). Now fix an arbitrary $x \in \mathcal{R}$. Let $\tau_{[x,+\infty)}$ denote the first time when ξ enters $[x,+\infty)$. (2.15) implies that $\xi^+_{T_x^+-} = x$ $\mathbf{P}_{0,\theta}$-a.s. on the event $\{\xi_{\tau_x^+-} < x = \xi_{\tau_x^+}\}$, which in turn implies that $\tau_{[x,+\infty)} < \tau_x^+$ $\mathbf{P}_{0,\theta}$-a.s. on $\{\xi_{\tau_x^+-} < x = \xi_{\tau_x^+}\}$. Hence $\mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+-} < x = \xi_{\tau_x^+}\right) = 0$, otherwise $\mathbf{P}_{0,\theta}\left(\tau_{[x,+\infty)} < \tau_x^+\right) > 0$, which contradicts the upwards regularity of (ξ,Θ). Hence we complete the proof. □

We note that the result in Proposition 2.8 holds only for almost every $x > 0$. In the following we give sufficient conditions under which it holds for every $x > 0$.

Proposition 2.9. *Suppose the conditions in Proposition 2.8 hold. Let $(X,\mathbb{P})$ denote the ssMp underlying $((\xi,\Theta),\mathbf{P})$ via Lamperti-Kiu transform. If $(X,\mathbb{P})$ is a Feller process and $a^+(v) > 0$ for every $v \in \mathcal{S}$, then for every $\theta \in \mathcal{S}$ and every $x > 0$,*

$$\mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+-} < x \ \text{or}\ \Theta_{\tau_x^+-} \neq \Theta_{\tau_x^+}; \xi_{\tau_x^+} = x\right) = 0,$$

and for every bounded continuous function $g : \mathbb{R}^+ \times \mathcal{S} \times \mathcal{S} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, the function

$$x \mapsto \mathbf{P}_{0,\theta}\left[g(\tau_x^+, \Theta_{\tau_x^+-}, \Theta_{\tau_x^+}, x - \xi_{\tau_x^+-}, \xi_{\tau_x^+} - x)1_{\{\xi_{\tau_x^+}=x\}}\right]$$

is right continuous on $[0,+\infty)$. Moreover, the density function $u_\theta^+(\mathrm{d}v,x)$ of $U_\theta^+(\mathrm{d}v,\mathrm{d}x)$ can take a unique version such that $x \mapsto a^+(v)u_\theta^+(\mathrm{d}v,x)$ is right continuous on $(0,+\infty)$ in the sense of vague convergence. In this case, (2.18) holds for every $x > 0$ and every nonnegative measurable function $f : \mathcal{S} \times \mathcal{S} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$.

Proof. For every $(x,\theta) \in \mathbb{R}^+ \times \mathcal{S}$, let $p^\theta(x) := \mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+} = x\right)$, $p_1^\theta(x) := \mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+-} = \xi_{\tau_x^+} = x\right)$ and $p_2^\theta(x) := p^\theta(x) - p_1^\theta(x) = \mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+-} < x = \xi_{\tau_x^+}\right)$. By Proposition 2.8 we have $p_2^\theta(x) = 0$ for almost every $x > 0$. Since $a^+(v) > 0$ for all $v \in \mathcal{S}$, it follows by Proposition 2.7 and Lemma 2.1 that

$$\begin{aligned}\mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+} > x\right) &= \int_{\mathcal{S}\times[0,x]} \bar{\Pi}_v^+(x-z)U_\theta^+(\mathrm{d}v,\mathrm{d}z) \\ &= \int_0^x \mathrm{d}z \int_{\mathcal{S}} \bar{\Pi}_v^+(x-z)u_\theta^+(\mathrm{d}v,z).\end{aligned}$$

Here $\bar{\Pi}_v^+(u) = \Pi^+(v,\mathcal{S},(u,+\infty))$. Obviously from the above equation $x \mapsto p^\theta(x) = 1 - \mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+} > x\right)$ is right continuous on $[0,+\infty)$. Suppose $x_n, x \in \mathbb{R}^+$ and $x_n \downarrow x$. Since X is a Feller process, it follows by [17, Theorem 4.2.5] that

$$(X, \mathbb{P}_{\theta \mathrm{e}^{-x_n}}) \to (X, \mathbb{P}_{\theta \mathrm{e}^{-x}})$$

in distribution under the Skorokhod topology. For $n \geq 1$, let $(Y^{(n)}, \mathbb{P}^*)$ and $(Y, \mathbb{P}^*)$ be couplings of $(X, \mathbb{P}_{\theta \mathrm{e}^{-x_n}})$ and $(X, \mathbb{P}_{\theta \mathrm{e}^{-x}})$ respectively, such that $Y^{(n)} \to Y$ $\mathbb{P}^*$-a.s. in the Skorokhod topology. Let $\varsigma_0 := \inf\{t \geq 0: \|Y_t\| > 1\}$ and $\varsigma_n := \inf\{t \geq 0: \|Y_t^{(n)}\| > 1\}$ for $n \geq 0$. Since X is sphere-exterior regular, so is Y, which implies that $\|Y_t\| \neq 1$ for any $t < \varsigma_0$ $\mathbb{P}^*$-a.s. In view of this, it follows by [32, Theorem 13.6.4] that

$$(Y^{(n)}_{\varsigma_n -}, Y^{(n)}_{\varsigma_n}) \to (Y_{\varsigma_0 -}, Y_{\varsigma_0}) \quad \mathbb{P}^*\text{-a.s.}$$

as $n \to +\infty$. Hence $\left((X_{\tau_1^{\ominus}-}, X_{\tau_1^{\ominus}}), \mathbb{P}_{\theta \mathrm{e}^{-x_n}}\right)$ converges in distribution to $\left((X_{\tau_1^{\ominus}-}, X_{\tau_1^{\ominus}}), \mathbb{P}_{\theta \mathrm{e}^{-x}}\right)$. The weak convergence yields that

$$\begin{aligned}
p_1^\theta(x) &= \mathbf{P}_{-x,\theta}\left(\xi_{\tau_0^+ -} = \xi_{\tau_0^+} = 0\right) \\
&= \mathbb{P}_{\theta \mathrm{e}^{-x}}\left(X_{\tau_1^{\ominus}-} \in \mathbb{S}^{d-1}, X_{\tau_1^{\ominus}} \in \mathbb{S}^{d-1}\right) \\
&\geq \limsup_{n \to +\infty} \mathbb{P}_{\theta \mathrm{e}^{-x_n}}\left(X_{\tau_1^{\ominus}-} \in \mathbb{S}^{d-1}, X_{\tau_1^{\ominus}} \in \mathbb{S}^{d-1}\right) \\
&= \limsup_{n \to +\infty} p_1^\theta(x_n).
\end{aligned}$$

This and the right continuity of $p^\theta(\cdot)$ imply that $\liminf_{n \to +\infty} p_2^\theta(x_n) \geq p_2^\theta(x)$. Hence

$$p_2^\theta(x) = \mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+ -} < x = \xi_{\tau_x^+}\right) = 0 \quad \forall x > 0. \tag{2.19}$$

It then follows by Lemma 2.2 that

$$\mathbf{P}_{0,\theta}\left(\Theta_{\tau_x^+ -} \neq \Theta_{\tau_x^+}, \xi_{\tau_x^+} = x\right) = 0, \quad \forall x > 0. \tag{2.20}$$

We need to show that

$$\begin{aligned}
&\lim_{n \to +\infty} \mathbf{P}_{0,\theta}\left[g(\tau_{x_n}^+, \Theta_{\tau_{x_n}^+ -}, \Theta_{\tau_{x_n}^+}, x_n - \xi_{\tau_{x_n}^+ -}, \xi_{\tau_{x_n}^+} - x_n)\mathbb{1}_{\{\xi_{\tau_{x_n}^+} = x_n\}}\right] \\
&\quad = \mathbf{P}_{0,\theta}\left[g(\tau_x^+, \Theta_{\tau_x^+ -}, \Theta_{\tau_x^+}, x - \xi_{\tau_x^+ -}, \xi_{\tau_x^+} - x)\mathbb{1}_{\{\xi_{\tau_x^+} = x\}}\right].
\end{aligned} \tag{2.21}$$

for any sequence $x_n, x \in \mathbb{R}^+$, $x_n \downarrow x$ and any bounded continuous function $g: \mathbb{R}^+ \times \mathcal{S} \times \mathcal{S} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. Let $A_n := \{\xi_{\tau_{x_n}^+} = x_n\}$ and $A := \{\xi_{\tau_x^+} = x\}$. By the strong Markov property and the fact that $\lim_{y \to 0+} p^v(y) = p^v(0) = 1$ for every $v \in \mathcal{S}$, we have for every $\theta \in \mathcal{S}$

$$\begin{aligned}
\mathbf{P}_{0,\theta}(A \setminus A_n) &= \mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+} = x, \xi_{\tau_{x_n}^+} > x_n\right) \\
&= \mathbf{P}_{0,\theta}\left(\mathbf{P}_{0,\Theta_{\tau_x^+}}\left(\xi_{\tau_{x_n - x}^+} > x_n - x\right); \xi_{\tau_x^+} = x\right) \\
&= \mathbf{P}_{0,\theta}\left[\left(1 - p^{\Theta_{\tau_x^+}}(x_n - x)\right)\mathbb{1}_{\{\xi_{\tau_x^+} = x\}}\right] \\
&\to 0, \quad \text{as } n \to +\infty.
\end{aligned}$$

Since $\mathbf{P}_{0,\theta}(A_n \setminus A) - \mathbf{P}_{0,\theta}(A \setminus A_n) = \mathbf{P}_{0,\theta}(A_n) - \mathbf{P}_{0,\theta}(A) = p^\theta(x_n) - p^\theta(x) \to 0$ as $n \to +\infty$, we have

$$\mathbf{P}_{0,\theta}(A \Delta A_n) = \mathbf{P}_{0,\theta}(A_n \setminus A) + \mathbf{P}_{0,\theta}(A \setminus A_n) \to 0 \quad \text{as } n \to +\infty. \tag{2.22}$$

Note that by (2.19) and (2.20)

$$
\begin{aligned}
&\Big|\mathbf{P}_{0,\theta}\left[g(\tau^+_{x_n},\Theta_{\tau^+_{x_n}-},\Theta_{\tau^+_{x_n}},x_n-\xi_{\tau^+_{x_n}-},\xi_{\tau^+_{x_n}}-x_n)\mathbb{1}_{\{\xi_{\tau^+_{x_n}}=x_n\}}\right]\\
&-\mathbf{P}_{0,\theta}\left[g(\tau^+_x,\Theta_{\tau^+_x-},\Theta_{\tau^+_x},x-\xi_{\tau^+_x-},\xi_{\tau^+_x}-x)\mathbb{1}_{\{\xi_{\tau^+_x}=x\}}\right]\Big|\\
&=\Big|\mathbf{P}_{0,\theta}\left[g(\Theta_{\tau^+_{x_n}},\Theta_{\tau^+_{x_n}},x_n-\xi_{\tau^+_{x_n}},\xi_{\tau^+_{x_n}}-x_n)\mathbb{1}_{\{\xi_{\tau^+_{x_n}}=x_n\}}\right]\\
&-\mathbf{P}_{0,\theta}\left[g(\Theta_{\tau^+_x},\Theta_{\tau^+_x},x-\xi_{\tau^+_x},\xi_{\tau^+_x}-x)\mathbb{1}_{\{\xi_{\tau^+_x}=x\}}\right]\Big|\\
&\le\Big||\mathbf{P}_{0,\theta}\left[g(\tau^+_{x_n},\Theta_{\tau^+_{x_n}},\Theta_{\tau^+_{x_n}},x_n-\xi_{\tau^+_{x_n}},\xi_{\tau^+_{x_n}}-x_n)\left(\mathbb{1}_{\{\xi_{\tau^+_{x_n}}=x_n\}}-\mathbb{1}_{\{\xi_{\tau^+_x}=x\}}\right)\right]\Big|\\
&+\Big||\mathbf{P}_{0,\theta}\left[\left(g(\tau^+_{x_n},\Theta_{\tau^+_{x_n}},\Theta_{\tau^+_{x_n}},x_n-\xi_{\tau^+_{x_n}},\xi_{\tau^+_{x_n}}-x_n)-g(\tau^+_x,\Theta_{\tau^+_x},\Theta_{\tau^+_x},x-\xi_{\tau^+_x},\xi_{\tau^+_x}-x)\right)\mathbb{1}_{\{\xi_{\tau^+_x}=x\}}\right]\Big|\\
&\le\|g\|_\infty\mathbf{P}_{0,\theta}\left(A\Delta A_n\right)\\
&+\mathbf{P}_{0,\theta}\left[\Big|g(\tau^+_{x_n},\Theta_{\tau^+_{x_n}},\Theta_{\tau^+_{x_n}},x_n-\xi_{\tau^+_{x_n}},\xi_{\tau^+_{x_n}}-x_n)-g(\tau^+_x,\Theta_{\tau^+_x},\Theta_{\tau^+_x},x-\xi_{\tau^+_x},\xi_{\tau^+_x}-x)\Big|\right].
\end{aligned}
$$

We have $\tau^+_{x_n}\downarrow\tau^+_x$ $\mathbf{P}_{0,\theta}$-a.s. by the upwards regularity of (ξ,Θ) and hence $\left(\Theta_{\tau^+_{x_n}},\xi_{\tau^+_{x_n}}\right)\to\left(\Theta_{\tau^+_x},\xi_{\tau^+_x}\right)$ $\mathbf{P}_{0,\theta}$-a.s. by the right continuity of (ξ,Θ). In view of this and (2.22), (2.21) follows by letting $n\to+\infty$ in the above inequality.

By (2.19) and (2.20), we have for every $x>0$ and every nonnegative measurable function $f:\mathcal{S}\times\mathcal{S}\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$,

$$
\begin{aligned}
\mathbf{P}_{0,\theta}\left[f(\Theta_{\tau^+_x-},\Theta_{\tau^+_x},x-\xi_{\tau^+_x-},\xi_{\tau^+_x}-x)\mathbb{1}_{\{\xi_{\tau^+_x}=x\}}\right] &= \mathbf{P}_{0,\theta}\left[f(\Theta_{\tau^+_x},\Theta_{\tau^+_x},0,0)\mathbb{1}_{\{\xi_{\tau^+_x}=x\}}\right]\\
&= \int_{\mathcal{S}} f(v,v,0,0)\mathbf{P}_{0,\theta}\left(\Theta_{\tau^+_x}\in \mathrm{d}v,\xi_{\tau^+_x}=x\right).
\end{aligned}
$$

In view of this and Proposition 2.8, we can set the density function $u^+_\theta(\mathrm{d}v,x)$ of $U^+_\theta(\mathrm{d}v,\mathrm{d}x)$ to be $\frac{1}{a^+(v)}\mathbf{P}_{0,\theta}\left(\Theta_{\tau^+_x}\in\mathrm{d}v,\xi_{\tau^+_x}=x\right)$ for every $x>0$, in which case, $x\mapsto a^+(v)u^+_\theta(\mathrm{d}v,x)=\mathbf{P}_{0,\theta}\left(\Theta_{\tau^+_x}\in\mathrm{d}v,\xi_{\tau^+_x}=x\right)$ is right continuous on $(0,+\infty)$ in the sense of vague convergence, because $x\mapsto\mathbf{P}_{0,\theta}\left[h(\Theta_{\tau^+_x});\xi_{\tau^+_x}=x\right]$ is right continuous on $(0,+\infty)$ for every bounded continuous function $h:\mathcal{S}\to\mathbb{R}$. $\square$

2.3. *Long time behavior of MAP.* It is well-known that for any $\mathbb{R}$-valued Lévy process χ one has $\chi_t/t\to\mathrm{E}\chi_1$ almost surely whenever $\mathrm{E}\chi_1$ is well-defined. Its proof relies on the classical strong law of large numbers. Following this, a Lévy process exhibits exactly one of the following behaviors: $\lim_{t\to+\infty}\chi_t=+\infty$ a.s., $\lim_{t\to+\infty}\chi_t=-\infty$ a.s. and $\limsup_{t\to+\infty}\chi_t=-\liminf_{t\to+\infty}\chi_t=+\infty$ a.s. according as $\mathrm{E}\chi_1>,<,=0$. This basic trichotomy is also true for the MAPs where $(\Theta_t)_{t\ge0}$ is a positive recurrent Markov process on a countable state space. We refer to [3] and the references therein. In such case, let $\tau_0(i):=0$ and $\{\tau_n(i):n\ge1\}$ denote the renewal sequence of successive return times to each state $i\in\mathcal{S}$. Then for each i, $\{\xi_{\tau_n(i)}:n\ge0\}$ constitutes an ordinary random walk. In fact, a law of large numbers can be obtained by applying known results for these embedded random walks, but with considerable additional analysis. Regarding the more general situation when the modulator Θ has an uncountably infinite state space, we note that a natural substitute for $\{\tau_n(i) : n \geq 1\}$ is a sequence of random times $\{R_n : n \geq 0\}$, in terms of which the process can be decomposed into independent and stationary blocks. In order to construct such random times, we assume the MAP satisfies the following Harris-type condition: There exist a constant $\delta > 0$, a probability measure ρ on $\mathcal{S}$ and a family of measures $\{\phi(\theta, \cdot) : \theta \in \mathcal{S}\}$ on $\mathbb{R}$ with $\inf_{\theta\in\mathcal{S}} \phi(\theta, \mathbb{R}) > 0$ such that

$$\text{(HT)} \qquad \mathbf{P}_{0,\theta}\left(\xi_\delta \in \Gamma,\ \Theta_\delta \in A\right) \geq \phi(\theta,\Gamma)\rho(A) \quad \forall \theta \in \mathcal{S},\ A \in \mathcal{B}(\mathcal{S}),\ \Gamma \in \mathcal{B}(\mathbb{R}).$$

This section aims at providing the trichotomy regarding the almost sure behavior of ξ_t as $t \to +\infty$ when condition (HT) is satisfied.

Define $M_0 := \Theta_0$, $S_0 := \xi_0$ and for any $n \geq 1$, define

$$M_n := \Theta_{n\delta},\ \Delta_n := \xi_{n\delta} - \xi_{(n-1)\delta} \quad \text{and} \quad S_n := S_0 + \sum_{k=1}^{n} \Delta_k.$$

It is easy to verify that $((S_n, M_n)_{n\geq 0}, \mathbf{P})$ is a discrete-time MAP satisfying

$$\mathbf{P}_{0,\theta}\left(\Delta_1 \in \Gamma,\ M_1 \in A\right) \geq \phi(\theta,\Gamma)\rho(A) \tag{2.23}$$

for all $\theta \in \mathcal{S}, A \in \mathcal{B}(\mathcal{S})$ and $\Gamma \in \mathcal{B}(\mathbb{R})$. In particular we have

$$\mathbf{P}_{0,\theta}\left(M_1 \in A\right) \geq \epsilon\rho(A) \quad \forall \theta \in \mathcal{S},\ A \in \mathcal{B}(\mathcal{S}),$$

where $\epsilon := \inf_{\theta\in\mathcal{S}} \phi(\theta, \mathbb{R}) > 0$. This implies that $\{M_n : n \geq 0\}$ is an irreducible and strongly aperiodic Harris recurrent chain on $\mathcal{S}$. Given this and (2.23), it follows by [28, 29] that there exists a sequence of regeneration times $0 \leq R_0 < R_1 < \cdots < +\infty$ such that $\{R_{n+1} - R_n : n \geq 0\}$ is a sequence of independent and identically distributed nonnegative random variables, and that the random blocks $\{M_{R_n}, \cdots, M_{R_{n+1}-1}, \Delta_{R_n+1}, \cdots, \Delta_{R_{n+1}}\}$ are independent with

$$\mathbf{P}_{0,\theta}\left[M_{R_n} \in A \,|\, \mathcal{G}_{R_n-1}, \Delta_{R_n}\right] = \rho(A) \quad \forall A \in \mathcal{B}(\mathcal{S}),$$

where $\mathcal{G}_k$ denotes the σ-field generated by $\{M_0, \cdots, M_k, \Delta_1, \cdots, \Delta_k\}$.

We assume that $(\Theta_t)_{t\geq 0}$ has an invariant distribution π. By [4, Theorem 3.2] π is uniquely determined by

$$\pi(A) = \frac{1}{\mathbf{P}_{0,\rho}[R_1]} \mathbf{P}_{0,\rho}\left[\sum_{j=0}^{R_1-1} \mathbb{1}_{\{M_j \in A\}}\right] \quad \forall A \in \mathcal{B}(\mathcal{S}) \tag{2.24}$$

where $0 < \mathbf{P}_{0,\rho}[R_1] < +\infty$. It follows that

$$\begin{aligned}
\mathbf{P}_{0,\pi}\left[S_1\right] &= \frac{1}{\mathbf{P}_{0,\rho}[R_1]} \sum_{j=0}^{+\infty} \int_{\mathcal{S}} \mathbf{P}_{0,\rho}\left[S_{j+1} - S_j \,|\, M_j = \theta\right] \mathbf{P}_{0,\rho}\left(M_j \in \mathrm{d}\theta,\ j \leq R_1 - 1\right) \\
&= \frac{1}{\mathbf{P}_{0,\rho}[R_1]} \mathbf{P}_{0,\rho}\left[\sum_{j=0}^{R_1-1} (S_{j+1} - S_j)\right] \\
&= \frac{1}{\mathbf{P}_{0,\rho}[R_1]} \mathbf{P}_{0,\rho}\left[S_{R_1}\right], \qquad (2.25)
\end{aligned}$$

whenever $\mathbf{P}_{0,\pi}\left[|S_1|\right] < +\infty$. The regeneration structure implies that $\left(S_{R_{n+1}} - S_{R_n}\right)$ is independent of $\{S_k, k \le R_n\}$, and its distribution is independent of n. Let $N_n := \sup\{k : \; R_k \le n\}$. We can write

$$S_n = S_{R_0 \wedge n} + \left[(S_{R_1} - S_{R_0}) + \cdots + \left(S_{R_{N_n}} - S_{R_{N_n - 1}}\right)\right] + \left(S_n - S_{R_{N_n}}\right).$$

It is easy to see that $S_{R_0\wedge n}/n \to 0$ almost surely since R_0 is finite and $\lim_{n\to+\infty} S_{R_0\wedge n} = S_{R_0} < +\infty$ almost surely. Note that $(S_{R_1} - S_{R_0}) + \cdots + \left(S_{R_{N_n}} - S_{R_{N_n-1}}\right)$ is a random sum of i.i.d summands. In view of (2.25), we have by the standard LLN and the elementary renewal theory that

$$\begin{aligned}
& \lim_{n\to+\infty} \frac{(S_{R_1} - S_{R_0}) + \cdots + \left(S_{R_{N_n}} - S_{R_{N_n-1}}\right)}{n} \\
= & \lim_{n\to+\infty} \frac{(S_{R_1} - S_{R_0}) + \cdots + \left(S_{R_{N_n}} - S_{R_{N_n-1}}\right)}{N_n} \cdot \frac{N_n}{n} \\
= & \; \mathbf{P}_{0,\rho}\left[S_{R_1}\right] \cdot \frac{1}{\mathbf{P}_{0,\rho}[R_1]} = \mathbf{P}_{0,\pi}\left[S_1\right] \quad \mathbf{P}_{0,\theta}\text{-a.s.}
\end{aligned}$$

Moreover one can easily show by Borel-Cantelli lemma that $\left(S_n - S_{R_{N_n}}\right)/n \to 0$ $\mathbf{P}_{0,\theta}$-a.s. if $\mathbf{P}_{0,\rho}\left[\max_{1\le k\le R_1} |S_k|\right] < +\infty$. We have hence proved the following lemma.

LEMMA 2.3. *If* $\mathbf{P}_{0,\rho}\left[\max_{1\le k\le R_1}|S_k|\right] < +\infty$, *then* $S_n/n \to \mathbf{P}_{0,\pi}[S_1]$ $\mathbf{P}_{0,\theta}$-*a.s. for every* $\theta \in \mathcal{S}$.

LEMMA 2.4. *If* $\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,t]}|\xi_s|\right]$ *is finite for some* $t > 0$, *then it is finite for all* $t > 0$ *and* $\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,e_q]}|\xi_s|\right]$ *is finite for all* $q > 0$.

PROOF. In this proof we use $\|\xi\|_t$ to denote $\sup_{s\in[0,t]}|\xi_s|$. Let $f(t) := \mathbf{P}_{0,\pi}\left[\|\xi\|_t\right]$ for $t \ge 0$. We observe that for any $t, r > 0$,

$$\|\xi\|_{t+r} \le \|\xi\|_t \vee \left(\sup_{s\in[t,t+r]}|\xi_s - \xi_t| + |\xi_t|\right) \le \|\xi\|_t + \sup_{s\in[t,t+r]}|\xi_s - \xi_t|. \tag{2.26}$$

By the Markov property and translation invariance in ξ, we have

$$\mathbf{P}_{0,\pi}\left[\sup_{s\in[t,t+r]}|\xi_s - \xi_t|\right] = \mathbf{P}_{0,\pi}\left[\mathbf{P}_{0,\Theta_t}\left[\|\xi\|_r\right]\right] = \mathbf{P}_{0,\pi}\left[\|\xi\|_r\right] = f(r).$$

The second equality is because π is an invariant distribution of $(\Theta_t)_{t\ge 0}$. Hence by (2.26) we get $f(t+r) \le f(t) + f(r)$. Given that $f(t)$ is finite for some $t > 0$, f is a nonnegative locally bounded subadditive function on $[0, +\infty)$. Hence there exist some constants $b, c > 0$ such that $f(t) \le ct + b$ for all $t > 0$. Consequently, $\mathbf{P}_{0,\pi}\left[\|\xi\|_{e_q}\right] = \int_0^{+\infty} q\mathrm{e}^{-qt} f(t)\mathrm{d}t < +\infty$ for all $q > 0$. □

PROPOSITION 2.10. *Suppose $((\xi,\Theta),\mathbf{P})$ is a MAP satisfying (HT) and π is an invariant distribution for $(\Theta_t)_{t\geq 0}$. If*

$$\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,1]}|\xi_s|\right]<+\infty \tag{2.27}$$

then $\xi_t/t\to\mathbf{P}_{0,\pi}[\xi_1]$ $\mathbf{P}_{0,\theta}$-a.s. for every $\theta\in\mathcal{S}$.

PROOF. Without loss of generality we may and do assume that (HT) holds for $\delta=1$. This proof works through for any $\delta>0$ with minor modifications. By Lemma 2.4, condition (2.27) implies that $\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,t]}|\xi_s|\right]<+\infty$ for all $t>0$ and $\mathbf{P}_{0,\pi}[|\Delta_1|]=\mathbf{P}_{0,\pi}[|\xi_1|]<+\infty$. We have

$$\begin{aligned}
\mathbf{P}_{0,\rho}\left[\max_{1\leq k\leq R_1}|S_k|\right] &\leq \mathbf{P}_{0,\rho}\left[\sum_{j=0}^{R_1-1}|\Delta_{j+1}|\right]\\
&= \sum_{j=0}^{+\infty}\int_{\mathcal{S}}\mathbf{P}_{0,\rho}\left[|\Delta_{j+1}|\mid M_j=\theta\right]\mathbf{P}_{0,\rho}\left(M_j\in\mathrm{d}\theta, j\leq R_1-1\right)\\
&= \int_{\mathcal{S}}\mathbf{P}_{0,\theta}\left[|\Delta_1|\right]\mathbf{P}_{0,\rho}\left[\sum_{j=0}^{R_1-1}\mathbb{1}_{\{M_j\in\mathrm{d}\theta\}}\right]\\
&= \mathbf{P}_{0,\rho}[R_1]\mathbf{P}_{0,\pi}[|\Delta_1|]<+\infty,
\end{aligned}\tag{2.28}$$

where in the last equality we use (2.24). It follows by Lemma 2.3 that $S_n/n\to\mathbf{P}_{0,\pi}[S_1]=\mathbf{P}_{0,\pi}[\xi_1]$ $\mathbf{P}_{0,\theta}$-a.s. for every $\theta\in\mathcal{S}$. Note that for any $t\in[R_k,R_{k+1})$,

$$\frac{S_{R_k}}{R_k}\frac{R_k}{R_{k+1}}-\frac{\sup_{s\in[R_k,R_{k+1}]}|\xi_s-S_{R_k}|}{R_{k+1}}\leq\frac{\xi_t}{t}\leq\frac{S_{R_k}}{R_k}+\frac{\sup_{s\in[R_k,R_{k+1}]}|\xi_s-S_{R_k}|}{R_k}.$$

It is known by the renewal theorem that $R_k/k\to\mathbf{P}_{0,\rho}[R_1]$ $\mathbf{P}_{0,\theta}$-a.s. Hence to prove $\xi_t/t\to\mathbf{P}_{0,\pi}[\xi_1]$ $\mathbf{P}_{0,\theta}$-a.s., it suffices to prove that

$$\frac{\sup_{s\in[R_k,R_{k+1}]}|\xi_s-S_{R_k}|}{k}\to 0\quad\text{as } k\to+\infty\quad \mathbf{P}_{0,\theta}\text{-a.s.} \tag{2.29}$$

for every $\theta\in\mathcal{S}$. The regeneration structure implies that $\{\sup_{s\in[R_k,R_{k+1}]}|\xi_s-S_{R_k}|:k\geq 1\}$ under $\mathbf{P}_{0,\theta}$ is a family of i.i.d. random variables which have the same distribution as $\left(\sup_{s\in[0,R_1]}|\xi_s|,\mathbf{P}_{0,\rho}\right)$. Hence by the second Borel-Cantelli lemma, (2.29) holds if and only if

$$\mathbf{P}_{0,\rho}\left[\sup_{s\in[0,R_1]}|\xi_s|\right]<+\infty. \tag{2.30}$$

We note that

$$\sup_{s\in[0,R_1]}|\xi_s|\leq\max_{0\leq k\leq R_1-1}|S_k|+\max_{0\leq k\leq R_1-1}\sup_{s\in[k,k+1]}|\xi_s-S_k|.$$

Applying similar calculation as in (2.28) we can deduce that

$$\begin{aligned}\mathbf{P}_{0,\rho}\left[\max_{0\le k\le R_1-1}\sup_{s\in[k,k+1]}|\xi_s-S_k|\right] &\le \mathbf{P}_{0,\rho}\left[\sum_{k=0}^{R_1-1}\sup_{s\in[k,k+1]}|\xi_s-S_k|\right]\\ &= \mathbf{P}_{0,\rho}[R_1]\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,1]}|\xi_s|\right]<+\infty.\end{aligned}$$

Hence (2.30) follows from this and (2.28), completing the proof. □

PROPOSITION 2.11. *Suppose the conditions of Proposition 2.10 hold. Then we have (a') $\xi_t\to+\infty$, (b') $\limsup_{t\to+\infty}\xi_t=+\infty$, $\liminf_{t\to+\infty}\xi_t=-\infty$ and (c') $\xi_t\to-\infty$ $\mathbf{P}_{0,\theta}$-a.s. for every $\theta\in\mathcal{S}$ according as (a) $\mathbf{P}_{0,\pi}[\xi_1]>0$, (b) $\mathbf{P}_{0,\pi}[\xi_1]=0$ and the increment distribution in each block is not concentrated at 0 and (c) $\mathbf{P}_{0,\pi}[\xi_1]<0$.*

PROOF. It is immediate from Proposition 2.10 that (a)⇒(a') and (c)⇒(c'). In case (b), we consider the sequence $\{S_{R_k}:k\ge 0\}$ which is a discrete-time random walk with mean 0 and the increment distribution not concentrated at 0. Hence $\limsup_{k\to+\infty}S_{R_k}=+\infty$ and $\liminf_{k\to+\infty}S_{R_k}=-\infty$ which implies (b'). □

REMARK 2.12. Let us make a brief remark on the condition (HT). This condition is of course not the most general condition under which the results of Propositions 2.10 and 2.11 hold. We believe an extension is possible, at least to some extent. One direction is to assume Harris recurrence of $(M_n)_{n\ge0}$ alone. However, in this way, instead of having i.i.d increments, $\{S_{R_n}:n\ge0\}$ has 1-dependent and stationary increments. Therefore in all places where we apply results for ordinary random walks, extensions to the case of 1-dependent and stationary increments are needed. Since this can not be done shortly, we have restricted this section to the case when condition (HT) is satisfied.

Hereafter we say that ξ_t *drifts to* $+\infty$, *oscillates* or *drifts to* $-\infty$ *at* θ, respectively, if $\lim_{t\to+\infty}\xi_t=+\infty$, $\limsup_{t\to+\infty}\xi_t=-\liminf_{t\to+\infty}\xi_t=+\infty$ or $\lim_{t\to+\infty}\xi_t=-\infty$ $\mathbf{P}_{0,\theta}$-a.s.

PROPOSITION 2.13. *For every $\theta\in\mathcal{S}$,*

$$\int_{\mathcal{S}\times\mathbb{R}^+}\mathrm{n}_v^+(\zeta=+\infty)U_\theta^+(\mathrm{d}v,\mathrm{d}z)=\begin{cases}0 & \text{if } \xi_t \text{ oscillates or drifts to } +\infty \text{ at } \theta,\\ 1 & \text{if } \xi_t \text{ drifts to } -\infty \text{ at } \theta.\end{cases}$$

PROOF. Let $\bar g_\infty$ denote the last time when ξ_t attains its running maximum. If ξ_t oscillates or drifts to $+\infty$ at θ, then $\mathbf{P}_{0,\theta}\left(\bar g_\infty=+\infty\right)=1$. By Proposition 2.3 we have

(2.31)
$$\mathbf{P}_{0,\theta}\left[\mathrm{e}^{-\lambda\bar g_{e_q}}\right]=\int_{\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^+}\mathrm{e}^{-\lambda r-qr}\left(q\ell^+(v)+\mathrm{n}_v^+\left(1-\mathrm{e}^{-q\zeta}\right)\right)V_\theta^+(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z)\quad\forall\lambda,q>0.$$

Letting $q\to0+$, we get by Fatou's lemma that

$$0=\mathbf{P}_{0,\theta}\left[\mathrm{e}^{-\lambda\bar g_\infty}\right]\ge\int_{\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^+}\mathrm{e}^{-\lambda r}\mathrm{n}_v^+(\zeta=+\infty)V_\theta^+(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z).$$

Then by letting $\lambda \to 0+$, we get by the monotone convergence theorem that

$$\int_{\mathcal{S}\times\mathbb{R}^+} \mathrm{n}_v^+(\zeta = +\infty) U_\theta^+(\mathrm{d}v, \mathrm{d}z) = 0.$$

On the other hand, if ξ_t drifts to $-\infty$ at θ, then $\mathbf{P}_{0,\theta}\left(\bar{g}_\infty < +\infty\right) = 1$. Note that for any $0 < q < \lambda/2$, the integrand in the right-hand side of (2.31) is bounded from above by $\mathrm{e}^{-\lambda r}\left(\frac{\lambda}{2}\ell^+(v) + \mathrm{n}_v^+\left(1 - \mathrm{e}^{-\lambda\zeta/2}\right)\right)$ and

$$\int_{\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^+} \mathrm{e}^{-\lambda r}\left(\frac{\lambda}{2}\ell^+(v) + \mathrm{n}_v^+\left(1 - \mathrm{e}^{-\lambda\zeta/2}\right)\right) V_\theta^+(\mathrm{d}r, \mathrm{d}v, \mathrm{d}z) = \mathbf{P}_{0,\theta}\left[\mathrm{e}^{-\frac{\lambda}{2}\bar{g}_{\mathrm{e}_{\lambda/2}}}\right] < +\infty.$$

Hence by letting $q \to 0+$ in (2.31) and using the dominated convergence theorem in the right-hand side and the monotone convergence theorem in the left hand side we get

$$\mathbf{P}_{0,\theta}\left[\mathrm{e}^{-\lambda\bar{g}_\infty}\right] = \int_{\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^+} \mathrm{e}^{-\lambda r}\mathrm{n}_v^+(\zeta = +\infty) V_\theta^+(\mathrm{d}r, \mathrm{d}v, \mathrm{d}z).$$

Letting $\lambda \to 0+$, we have

$$\int_{\mathcal{S}\times\mathbb{R}^+} \mathrm{n}_v^+(\zeta = +\infty) U_\theta^+(\mathrm{d}v, \mathrm{d}z) = \mathbf{P}_{0,\theta}\left(\bar{g}_\infty < +\infty\right) = 1,$$

which completes the proof. □

2.4. *Invariant measures.*

PROPOSITION 2.14. *Suppose $((\xi, \Theta), \mathbf{P})$ is a MAP on $\mathbb{R}\times\mathcal{S}$ and ν is an invariant measure for the modulator Θ. Then the measure*

$$\nu^+(\cdot) := \mathbf{P}_{0,\nu}\left[\int_0^1 \mathbb{1}_{\{\Theta_s\in\cdot\}}\mathrm{d}\bar{L}_s\right] \tag{2.32}$$

is an invariant measure for the modulator Θ^+ of the ascending ladder height process $((\xi^+, \Theta^+), \mathbf{P})$. Moreover, ν^+ is finite if and only if $\mathbf{P}_{0,\nu}\left[\bar{L}_1\right] < +\infty$.

PROOF. It suffices to show that

$$\int_0^{+\infty} \mathrm{e}^{-\alpha s}\mathbf{P}_{0,\nu^+}\left[f(\Theta_s^+)\right]\mathrm{d}s = \frac{1}{\alpha}\int_{\mathcal{S}} f(\theta)\nu^+(d\theta) \tag{2.33}$$

for any $\alpha > 0$ and nonnegative measurable function $f : \mathcal{S} \to \mathbb{R}^+$. The left integral is equal to

$$\begin{aligned} \mathbf{P}_{0,\nu^+}\left[\int_0^{+\infty} \mathrm{e}^{-\alpha s} f(\Theta_s^+)\mathrm{d}s\right] &= \mathbf{P}_{0,\nu^+}\left[\int_0^{+\infty} \mathrm{e}^{-\alpha\bar{L}_s} f(\Theta_s)\mathrm{d}\bar{L}_s\right] \\ &= \mathbf{P}_{0,\nu}\left[\int_0^1 \mathbf{P}_{0,\Theta_r}\left[\int_0^{+\infty} \mathrm{e}^{-\alpha\bar{L}_s} f(\Theta_s)\mathrm{d}\bar{L}_s\right]\mathrm{d}\bar{L}_r\right]. \end{aligned} \tag{2.34}$$

Recall that $s \mapsto \bar{L}_s$ is an additive functional of $(\Theta_t, \bar{\xi}_t - \xi_t)_{t\geq 0}$. Hence the law of $(\bar{L}_t, \Theta_t)_{t\geq 0}$ under $\mathbf{P}_{x,\theta}$ does not depend on x. The right hand side of (2.34) is equal to

$$\begin{aligned}
& \mathbf{P}_{0,\nu}\left[\int_0^1 \mathbf{P}_{\xi_r,\Theta_r}\left[\int_0^{+\infty} \mathrm{e}^{-\alpha \bar{L}_s} f(\Theta_s)\mathrm{d}\bar{L}_s\right]\mathrm{d}\bar{L}_r\right] \\
= \; & \mathbf{P}_{0,\nu}\left[\int_0^1 \mathrm{d}\bar{L}_r \int_r^{+\infty} \mathrm{e}^{-\alpha\left(\bar{L}_s-\bar{L}_r\right)} f(\Theta_s)\mathrm{d}\bar{L}_s\right] \\
= \; & \mathbf{P}_{0,\nu}\left[\int_0^{+\infty} \mathrm{d}\bar{L}_s \mathrm{e}^{-\alpha\bar{L}_s} f(\Theta_s) \int_0^{1\wedge s} \mathrm{e}^{\alpha\bar{L}_r}\mathrm{d}\bar{L}_r\right] \\
= \; & \frac{1}{\alpha}\left[\mathbf{P}_{0,\nu}\left[\int_0^1 \mathrm{e}^{-\alpha\bar{L}_s} f(\Theta_s)\left(\mathrm{e}^{\alpha\bar{L}_s}-1\right)\mathrm{d}\bar{L}_s\right] + \mathbf{P}_{0,\nu}\left[\int_1^{+\infty} \mathrm{e}^{-\alpha\bar{L}_s} f(\Theta_s)\left(\mathrm{e}^{\alpha\bar{L}_1}-1\right)\mathrm{d}\bar{L}_s\right]\right] \\
= \; & \frac{1}{\alpha}\left[\mathbf{P}_{0,\nu}\left[\int_0^1 f(\Theta_s)\mathrm{d}\bar{L}_s\right] - \mathbf{P}_{0,\nu}\left[\int_0^{+\infty} \mathrm{e}^{-\alpha\bar{L}_s} f(\Theta_s)\mathrm{d}\bar{L}_s\right]\right. \\
& \left. + \mathbf{P}_{0,\nu}\left[\int_1^{+\infty} \mathrm{e}^{-\alpha\left(\bar{L}_s-\bar{L}_1\right)} f(\Theta_s)\mathrm{d}\bar{L}_s\right]\right] \qquad (2.35)
\end{aligned}$$

In the first equality we use the Markov property and the additivity of $\bar{L}_s$. Using these facts again we have

$$\begin{aligned}
\mathbf{P}_{0,\nu}\left[\int_1^{+\infty} \mathrm{e}^{-\alpha\left(\bar{L}_s-\bar{L}_1\right)} f(\Theta_s)\mathrm{d}\bar{L}_s\right] &= \mathbf{P}_{0,\nu}\left[\mathbf{P}_{\xi_1,\Theta_1}\left[\int_0^{+\infty} \mathrm{e}^{-\alpha\bar{L}_r} f(\Theta_r)\mathrm{d}\bar{L}_r\right]\right] \\
&= \mathbf{P}_{0,\nu}\left[\mathbf{P}_{0,\Theta_1}\left[\int_0^{+\infty} \mathrm{e}^{-\alpha\bar{L}_r} f(\Theta_r)\mathrm{d}\bar{L}_r\right]\right] \\
&= \mathbf{P}_{0,\nu}\left[\int_0^{+\infty} \mathrm{e}^{-\alpha\bar{L}_r} f(\Theta_r)\mathrm{d}\bar{L}_r\right]. \qquad (2.36)
\end{aligned}$$

In the last equality we use the fact that $\mathbf{P}_{0,\nu}\left(\Theta_1 \in \cdot\right) = \nu(\cdot)$. In view of (2.36), the right hand side of (2.35) equals

$$\frac{1}{\alpha}\mathbf{P}_{0,\nu}\left[\int_0^1 f(\Theta_s)\mathrm{d}\bar{L}_s\right] = \frac{1}{\alpha}\int_{\mathcal{S}} f(\theta)\nu^+(d\theta).$$

Hence we get (2.33). □

Corollary 2.15. *Suppose the modulator Θ of $((\xi,\Theta),\mathbf{P})$ has an invariant distribution π. If $\mathbf{P}_{0,\pi}\left[\bar{L}_1\right] > 0$ and $\inf_{\theta\in\mathcal{S}}\left[\ell^+(\theta) + \mathrm{n}^+_\theta\left(1-\mathrm{e}^{-\zeta}\right)\right] > 0$, then the measure π^+ defined by*

$$\pi^+(\cdot) := \frac{1}{\mathbf{P}_{0,\pi}\left[\bar{L}_1\right]}\mathbf{P}_{0,\pi}\left[\int_0^1 \mathbb{1}_{\{\Theta_s\in\cdot\}}\mathrm{d}\bar{L}_s\right]$$

is an invariant distribution for the modulator Θ^+ of $((\xi^+,\Theta^+),\mathbf{P})$.

Proof. By Proposition 2.14, it suffices to show that $\mathbf{P}_{0,\pi}\left[\bar{L}_1\right] < +\infty$. Let

$$c := \inf_{\theta\in\mathcal{S}}\left[\ell^+(\theta) + \mathrm{n}^+_\theta\left(1-\mathrm{e}^{-\zeta}\right)\right] \in (0,+\infty).$$

By (2.3) and (2.4) we have

$$\begin{aligned}
\mathbf{P}_{0,\pi}\left[\bar{L}_1\right] &\leq \frac{1}{c}\mathbf{P}_{0,\pi}\left[\int_0^1 \ell^+(\Theta_s) + \mathrm{n}^+_{\Theta_s}\left(1-\mathrm{e}^{-\zeta}\right)\mathrm{d}\bar{L}_s\right] \\
&= \frac{1}{c}\left[\mathbf{P}_{0,\pi}\left[\int_0^1 \mathbb{1}_{\{s\in\bar{M}\}}\mathrm{d}s\right] + \mathbf{P}_{0,\pi}\left[\sum_{g_i\in\bar{G}, g_i\leq 1}\left(1-\mathrm{e}^{-\zeta^{(g_i)}}\right)\right]\right] \\
&\leq \frac{1}{c}\left[1 + \mathbf{P}_{0,\pi}\left[\sum_{g_i\in\bar{G}, g_i\leq 1}\left(1\wedge\zeta^{(g_i)}\right)\right]\right].
\end{aligned}$$

We note that among all the excursions that start in the time interval $[0,1]$, there is, at most, one excursion having a lifetime longer than 1, and the sum of lifetimes of other excursions does not exceed 1. Hence $\mathbf{P}_{0,\pi}\left[\sum_{g_i\in\bar{G}, g_i\leq 1}\left(1\wedge\zeta^{(g_i)}\right)\right]\leq 2$ and $\mathbf{P}_{0,\pi}\left[\bar{L}_1\right] < +\infty$. □

3. Duality. In this section we present the notion of duality as well as several results about duality. Here we suppose that E is a Polish space and μ is a σ-finite Radon measure on E. Suppose that $(X,\mathbb{P})$ and $(\hat{X},\mathbb{Q})$ are two, possibly killed, right continuous strong Markov processes having left limits in E except perhaps at their lifetime. We use ζ and $\hat{\zeta}$ respectively to denote their lifetimes. We take the convention that $0- = 0$.

DEFINITION 3.1. Two processes Markov processes $(X,\mathbb{P})$ and $(\hat{X},\mathbb{Q})$ are dual with respect to μ if for every bounded measurable functions $f,g : E\to\mathbb{R}$ and every $t\geq 0$,

$$\int_E \mu(\mathrm{d}x)g(x)\mathbb{P}_x[f(X_t), t<\zeta] = \int_E \mu(\mathrm{d}x)f(x)\mathbb{Q}_x[g(\hat{X}_t), t<\hat{\zeta}].$$

Note that there is no requirement that μ is a finite measure. The notion of duality is closely linked with reversibility. The following result is from [31, Theorem 2.1].

LEMMA 3.1. *Suppose that $(X,\mathbb{P})$ and $(\hat{X},\mathbb{Q})$ are dual with respect to μ, then,*

$$\int_E \mu(\mathrm{d}x)\mathbb{P}_x\left[F\left((X_s)_{s\leq t}\right)\mathbb{1}_{\{t<\zeta\}}\right] = \int_E \mu(\mathrm{d}x)\mathbb{Q}_x\left[F\left((\hat{X}_{(t-s)-})_{s\leq t}\right)\mathbb{1}_{\{t<\zeta\}}\right]$$

for every $t\geq 0$ and nonnegative functional $F : \mathbb{D}_E[0,t]\to\mathbb{R}^+$.

Finally, we present a result on the time reversal from the lifetime which can be found in [13, Theorem 13.34].

LEMMA 3.2. *Suppose that $(X,\mathbb{P})$ and $(\hat{X},\mathbb{Q})$ are dual with respect to μ. If the process X has initial distribution η and a finite lifetime ζ such that*

$$\int_E \mu(\mathrm{d}x)f(x) = \int_E \eta(\mathrm{d}x)\mathbb{P}_x\left[\int_0^\zeta f(X_t)\mathrm{d}t\right] \tag{3.1}$$

for every nonnegative measurable function $f : E\to\mathbb{R}$, then $\left((X_{(\zeta-t)-})_{0<t<\zeta}, \mathbb{P}_\eta\right)$ is a right continuous strong Markov process having the same transition rates as $(\hat{X},\mathbb{Q})$.

We remark here that in general the measure η appearing in (3.1) may not exist. If exists, it is uniquely determined by the reference measure μ, see, for example, [20, Theorem 2.12 and Section 6].

Throughout the remainder of this paper, we assume the process $((\xi,\Theta),\widetilde{\mathbf{P}})$ is a MAP with $\widetilde{\mathbf{P}}_{y,v}\left(\xi_0=y,\Theta_0=v\right)=1$ and is linked to $((\xi,\Theta),\mathbf{P})$ through the following weak reversability property: There exists a probability measure π on $\mathcal{S}$ with full support such that

$$\text{(WR)}\qquad \mathbf{P}_{0,\theta}(\xi_t\in \mathrm{d}z;\Theta_t\in\mathrm{d}\vartheta)\pi(\mathrm{d}\theta)=\widetilde{\mathbf{P}}_{0,\vartheta}(\xi_t\in\mathrm{d}z;\Theta_t\in\mathrm{d}\theta)\pi(\mathrm{d}\vartheta)\quad \forall t\geq 0.$$

By integrating (WR) over variable z, it follows that the Markov processes $((\Theta_t)_{t\geq 0},\{\mathbf{P}_{0,\theta},\theta\in\mathcal{S}\})$ and $((\Theta_t)_{t\geq 0},\{\widetilde{\mathbf{P}}_{0,\theta},\theta\in\mathcal{S}\})$ are dual with respect to the measure π. Hereafter we denote by $\hat{\mathbf{P}}_{x,\theta}$ the law of $(-\xi,\Theta)$ under $\widetilde{\mathbf{P}}_{-x,\theta}$. We will use the notation $\hat{}$ to specify the mathematical quantities related to the process $((\xi,\Theta),\hat{\mathbf{P}})$. In the following we give some examples for a MAP to be weakly reversible. Each example corresponds to a well-known class of ssMps via Lamperti-Kiu transform.

Example 3.1. Suppose $\mathcal{S}=\{s_1,\cdots,s_n\}$ is a finite set. It is known that the process $((\xi,\Theta),\mathbf{P})$ is a MAP on $\mathbb{R}\times\mathcal{S}$ if and only if $((\Theta_t)_{t\geq 0},\{\mathbf{P}_{x,\theta}:\theta\in\mathcal{S}\})$ is a (possibly killed) Markov chain on $\mathcal{S}$ whose law does not depend on x, and for each $s_j,s_k\in\mathcal{S}$ there exist a (non-killed) Lévy process ξ^j and an $\mathbb{R}$-valued random variable $\Xi_{j,k}$ such that when Θ is in state s_j, ξ evolves according to an independent copy of ξ^j, and when Θ changes from s_j to another state s_k, ξ has an additional jump which is an independent copy of $\Xi_{j,k}$ and until the next jump of Θ, ξ evolves according to an independent copy of ξ^k, and so on, until the lifetime of Θ. For such a MAP condition (WR) is equivalent to require that there is a MAP $((\xi,\Theta),\widetilde{\mathbf{P}})$ on $\mathbb{R}\times\mathcal{S}$ and a probability measure π on $\mathcal{S}$ such that $\pi_j=\pi(\{s_j\})>0$ for $1\leq j\leq n$ and

$$\text{(3.2)}\qquad \pi_j\widetilde{\mathbf{P}}_{0,s_j}\left[\mathrm{e}^{i\lambda\xi_t}\mathbb{1}_{\{\Theta_t=s_k\}}\right]=\pi_k\mathbf{P}_{0,s_k}\left[\mathrm{e}^{i\lambda\xi_t}\mathbb{1}_{\{\Theta_t=s_j\}}\right]\quad \forall t\geq 0,\ \lambda\in\mathbb{R},\ 1\leq j,k\leq n.$$

We let $(q_{j,k})_{1\leq j,k\leq n}$ denote the intensity matrix of the Markov chain Θ, $\psi_j(\lambda)$ denote the characteristic exponent of the Lévy process ξ^j and $J_{j,k}(\lambda)$ denote the characteristic function of the random variable $\Xi_{j,k}$. The matrix

$$\boldsymbol{F}(\lambda):=\mathrm{diag}(-\psi_1(\lambda),\cdots,-\psi_n(\lambda))+\left(q_{j,k}J_{j,k}(\lambda)\right)_{1\leq j,k\leq n}\quad \forall\lambda\in\mathbb{R}$$

is called the characteristic matrix exponent of the MAP $((\xi,\Theta),\mathbf{P})$ because

$$\mathbf{P}_{0,s_j}\left[\mathrm{e}^{i\lambda\xi_t}\mathbb{1}_{\{\Theta_t=s_k\}}\right]=\left(\mathrm{e}^{\boldsymbol{F}(\lambda)t}\right)_{j,k}\quad \forall t\geq 0,\ 1\leq j,k\leq n.$$

Equation (3.2), in terms of the characteristic matrix exponent, is equivalent to

$$\widetilde{\boldsymbol{F}}(\lambda)=\boldsymbol{\Delta}_\pi^{-1}F(\lambda)^{\mathrm{T}}\boldsymbol{\Delta}_\pi\quad \forall\lambda\in\mathbb{R},$$

where $\boldsymbol{\Delta}_\pi=\mathrm{diag}(\pi_1,\cdots,\pi_n)$. Condition (WR) is satisfied, in particular, if the process Θ is dual with itself with respect to a probability measure π and $\Xi_{j,k}\overset{d}{=}\Xi_{k,j}$ for all $1\leq j,k\leq n$, in which case we can take $\widetilde{\mathbf{P}}=\mathbf{P}$.

EXAMPLE 3.2. Suppose ∂ is an isolated extra state and the transition probabilities of $((\xi,\Theta),\mathbf{P})$ have the following form:

$$\begin{cases}\mathbf{P}_{x,\theta}\left(\xi_t\in \mathrm{d}y,\ \Theta_t\in \mathrm{d}v\right)=\mathrm{e}^{-\lambda t}\mathrm{P}^{\xi'}_{x}\left(\xi'_t\in \mathrm{d}y\right)\mathrm{P}^{\Theta'}_{\theta}\left(\Theta'_t\in \mathrm{d}v\right),\\ \mathbf{P}_{x,\theta}\left((\xi_t,\Theta_t)=\partial\right)=1-\mathrm{e}^{-\lambda t}\end{cases}$$

for all $t\ge 0$ and $(x,\theta)\in\mathbb{R}\times\mathcal{S}$, where $\lambda\ge 0$ is a constant, $(\xi',\mathrm{P}^{\xi'}_x)$ is a non-killed $\mathbb{R}$-valued Lévy process started from x and $(\Theta',\mathrm{P}^{\Theta'}_\theta)$ is a non-killed $\mathcal{S}$-valued Markov process started from θ. Then condition (WR) is satisfied if and only if there exists an $\mathcal{S}$-valued Markov process $((\Theta'_t)_{t\ge 0},\{\widetilde{\mathrm{P}}^{\Theta'}_\theta,\theta\in\mathcal{S}\})$, which is dual to $((\Theta'_t)_{t\ge 0},\{\mathrm{P}^{\Theta'}_\theta,\theta\in\mathcal{S}\})$ with respect to a probability measure π on $\mathcal{S}$. In this case, we can take the MAP $((\xi,\Theta),\widetilde{\mathbf{P}})$ to be such that its transition probabilities have the following form:

$$\begin{cases}\widetilde{\mathbf{P}}_{x,\theta}\left(\xi_t\in \mathrm{d}y,\ \Theta_t\in \mathrm{d}v\right)=\mathrm{e}^{-\lambda t}\mathrm{P}^{\xi'}_{x}\left(\xi'_t\in \mathrm{d}y\right)\widetilde{\mathrm{P}}^{\Theta'}_{\theta}\left(\Theta'_t\in \mathrm{d}v\right),\\ \widetilde{\mathbf{P}}_{x,\theta}\left((\xi_t,\Theta_t)=\partial\right)=1-\mathrm{e}^{-\lambda t}\end{cases}$$

for all $t\ge 0$ and $(x,\theta)\in\mathbb{R}\times\mathcal{S}$.

EXAMPLE 3.3. Suppose $\mathcal{S}=\mathbb{S}^{d-1}$ and for any orthogonal transformation $\mathcal{O}$ of $\mathbb{S}^{d-1}$ and $(x,\theta)\in\mathbb{R}\times\mathbb{S}^{d-1}$, $((\xi,\Theta),\mathbf{P}_{x,\theta})$ is equal in law with $\big((\xi,\mathcal{O}(\Theta)),\mathbf{P}_{x,\mathcal{O}^{-1}(\theta)}\big)$. In view of this property, if X is the ssMp associated with (ξ,Θ) by Lamperti-Kiu transform, then X is a rotationally invariant Markov process on $\mathbb{R}^d$. Hence its norm $(\|X_t\|)_{t\ge 0}$ is a positive ssMp, which in turn implies that ξ alone is a Lévy process. In this case condition (WR) is satisfied with $\widetilde{\mathbf{P}}=\mathbf{P}$ and π being the uniform measure on the sphere $\mathbb{S}^{d-1}$. We refer to [1, Proposition 3.2] for a proof.

PROPOSITION 3.2. *The processes* $((\xi,\Theta),\mathbf{P})$ *and* $((\xi,\Theta),\hat{\mathbf{P}})$ *are dual with respect to the measure* $\mathrm{Leb}\otimes\pi$*, where* Leb *is the Lebesgue measure on* $\mathbb{R}$.

PROOF. Suppose $f,g:\mathbb{R}\times\mathcal{S}\to\mathbb{R}^+$ are nonnegative measurable functions. By an application of Fubini's theorem, a change of variable and condition (WR) we get

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}x\pi(\mathrm{d}\theta)f(x,\theta)\mathbf{P}_{x,\theta}\left[g(\xi_t,\Theta_t)\right]\\
&\qquad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}x\pi(\mathrm{d}\theta)f(x,\theta)\mathbf{P}_{0,\theta}\left[g(x+\xi_t,\Theta_t)\right]\\
&\qquad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}y\pi(\mathrm{d}\theta)\mathbf{P}_{0,\theta}\left[f(y-\xi_t,\theta)g(y,\Theta_t)\right]\\
&\qquad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}y\pi(\mathrm{d}\theta)\int_{\mathbb{R}\times\mathcal{S}} \mathbf{P}_{0,\theta}\left(\xi_t\in\mathrm{d}z,\Theta_t\in\mathrm{d}\nu\right)f(y-z,\theta)g(y,\nu)\\
&\qquad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}y\pi(\mathrm{d}\nu)\int_{\mathbb{R}\times\mathcal{S}} \widetilde{\mathbf{P}}_{0,\nu}\left(\xi_t\in\mathrm{d}z,\Theta_t\in\mathrm{d}\theta\right)f(y-z,\theta)g(y,\nu)\\
&\qquad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}y\pi(\mathrm{d}\nu)g(y,\nu)\widetilde{\mathbf{P}}_{0,\nu}\left[f(y-\xi_t,\Theta_t)\right]\\
&\qquad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}y\pi(\mathrm{d}\nu)g(y,\nu)\hat{\mathbf{P}}_{0,\nu}\left[f(y+\xi_t,\Theta_t,)\right]\\
&\qquad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}y\pi(\mathrm{d}\nu)g(y,\nu)\hat{\mathbf{P}}_{y,\nu}\left[f(\xi_t,\Theta_t)\right]
\end{aligned}$$

for all $t\geq 0$. Hence we complete the proof. □

Lemma 3.3. *Suppose $t>0$. For every $x\in\mathbb{R}$, the process $(\xi_{(t-s)-}-\xi_t,\Theta_{(t-s)-})_{0\leq s\leq t}$ under $\mathbf{P}_{x,\pi}$ has the same law as $(\xi_s,\Theta_s)_{0\leq s\leq t}$ under $\hat{\mathbf{P}}_{0,\pi}$.*

Proof. In order to prove this lemma it suffices to consider the finite dimensional distributions. Let $n\geq 1$ be a fixed integer. For $0\leq k\leq n$ we take nonnegative measurable functions $f_k:\mathcal{S}\times\mathbb{R}\to\mathbb{R}^+$ and $0=t_0<t_1<t_2<\cdots<t_n<t_{n+1}=t$. Let $g:\mathbb{R}\to\mathbb{R}^+$ be a nonnegative measurable function. We know by Proposition 3.2 and Lemma 3.1 that the process $\left((\xi_{(t-s)-},\Theta_{(t-s)-})_{0\leq s\leq t},\mathbf{P}\right)$ has the same law as $\left((\xi_s,\Theta_s)_{0\leq s\leq t},\hat{\mathbf{P}}\right)$ both started according to the measure $\mathrm{Leb}\otimes\pi$. Using this and the quasi-left continuity of ξ, we have

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}x\pi(\mathrm{d}\theta)g(x)\mathbf{P}_{x,\theta}\left[f_0(\Theta_{(t-t_0)-},\xi_{(t-t_0)-}-\xi_t)\cdots f_n(\Theta_{(t-t_n)-},\xi_{(t-t_n)-}-\xi_t)\right]\\
&\quad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}x\pi(\mathrm{d}\theta)\mathbf{P}_{x,\theta}\left[f_0(\Theta_{(t-t_0)-},\xi_{(t-t_0)-}-\xi_{t-})\cdots f_n(\Theta_{(t-t_n)-},\xi_{(t-t_n)-}-\xi_{t-})g(\xi_{(t-t_{n+1})-})\right]\\
&\quad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}x\pi(\mathrm{d}\theta)\hat{\mathbf{P}}_{x,\theta}\left[f_0(\Theta_{t_0},\xi_{t_0}-\xi_0)\cdots f_n(\Theta_{t_n},\xi_{t_n}-\xi_0)g(\xi_{t_{n+1}})\right]\\
&\quad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}x\pi(\mathrm{d}\theta)\hat{\mathbf{P}}_{0,\theta}\left[f_0(\Theta_{t_0},\xi_{t_0})\cdots f_n(\Theta_{t_n},\xi_{t_n})g(\xi_{t_{n+1}}+x)\right]\\
&\quad= \hat{\mathbf{P}}_{0,\pi}\left[f_0(\Theta_{t_0},\xi_{t_0})\cdots f_n(\Theta_{t_n},\xi_{t_n})\int_{\mathbb{R}} g(\xi_{t_{n+1}}+x)\mathrm{d}x\right]\\
&\quad= \int_{\mathbb{R}\times\mathcal{S}} \mathrm{d}x\pi(\mathrm{d}\theta)g(x)\hat{\mathbf{P}}_{0,\theta}\left[f_0(\Theta_{t_0},\xi_{t_0})\cdots f_n(\Theta_{t_n},\xi_{t_n})\right],
\end{aligned}$$

where the last equality is obtained by Fubini's theorem and a change of variable. Since g is arbitrary, it follows by above equations that $\{(\xi_s, \Theta_s), 0 \le s \le t\}$ under $\hat{\mathbf{P}}_{0,\pi}$ has the same law as $\{(\xi_{(t-s)-} - \xi_t, \Theta_{(t-s)-}), 0 \le s \le t\}$ under $\mathbf{P}_{x,\pi}$ for almost every $x \in \mathbb{R}$. We observe that the law of the latter does not depend on x, thus the claim holds for every $x \in \mathbb{R}$. □

The upwards regularity of $((\xi,\Theta),\mathbf{P})$ implies that almost surely the local maxima of ξ during a finite time interval are all distinct. In view of this and Lemma 3.3, we have the following result.

PROPOSITION 3.3. *For every $t > 0$, $\left(\Theta_0, t - \bar{g}_t, \Theta_t, \bar{\xi}_t - \xi_t, \bar{g}_t, \bar{\Theta}_t, \bar{\xi}_t\right)$ under $\hat{\mathbf{P}}_{0,\pi}$ is equal in distribution to $\left(\Theta_t, \bar{g}_t, \Theta_0, \bar{\xi}_t, t - \bar{g}_t, \bar{\Theta}_t, \bar{\xi}_t - \xi_t\right)$ under $\mathbf{P}_{0,\pi}$.*

4. MAP conditioned to stay negative. In this section we assume that $((\xi,\Theta),\hat{\mathbf{P}})$ is an upwards regular MAP. Define

$$\hat{H}^+_\theta(y) := \hat{\mathbf{P}}_{y,\theta}\left(\tau_0^+ = +\infty\right), \quad \forall (y,\theta) \in \mathbb{R} \times \mathcal{S}.$$

Obviously $\hat{H}^+_\theta(y) = 0$ for all $y \ge 0$.

PROPOSITION 4.1. *Assume that*

$$\hat{\mathrm{n}}^+_v(\zeta = +\infty) > 0 \text{ for every } v \in \mathcal{S}, \tag{4.1}$$

then

(i) $\hat{H}^+_\theta(y) > 0$ for all $\theta \in \mathcal{S}$ and $y < 0$, and

(ii) $\hat{H}^+_{\Theta_t}(\xi_t)\mathbb{1}_{\{t<\tau_0^+\}}$ is a $\hat{\mathbf{P}}_{y,\theta}$-martingale for every $y < 0$ and $\theta \in \mathcal{S}$.

PROOF. (i) For $y < 0$ and $\theta \in \mathcal{S}$,

$$\hat{H}^+_\theta(y) = \hat{\mathbf{P}}_{0,\theta}\left(\tau^+_{-y} = +\infty\right) = \lim_{q\to 0+} \hat{\mathbf{P}}_{0,\theta}\left(\tau^+_{-y} > e_q\right).$$

It follows by Proposition 2.3 that

$$\begin{aligned}\hat{\mathbf{P}}_{0,\theta}\left(\tau^+_{-y} > e_q\right) &= \hat{\mathbf{P}}_{0,\theta}\left(\bar{\xi}_{e_q} \le -y\right) \\ &= \int_{\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^+} \mathrm{e}^{-qr}\mathbb{1}_{\{z\le -y\}}\left(q\hat{\ell}^+(v) + \hat{\mathrm{n}}^+_v\left(1-\mathrm{e}^{-q\zeta}\right)\right)\hat{V}^+_\theta(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z).\end{aligned}$$

Hence by condition (4.1) and Fatou's lemma,

$$\hat{H}^+_\theta(y) \ge \int_{\mathcal{S}\times[0,-y]} \hat{\mathrm{n}}^+_v(\zeta = +\infty)\hat{U}^+_\theta(\mathrm{d}v,\mathrm{d}z) > 0.$$

(ii) By the Markov property of $((\xi,\Theta),\hat{\mathbf{P}})$, we have for any $y < 0$ and $\theta \in \mathcal{S}$,

$$\begin{aligned}\hat{\mathbf{P}}_{y,\theta}\left[\hat{H}^+_{\Theta_t}(\xi_t)\mathbb{1}_{\{t<\tau_0^+\}}\right] &= \hat{\mathbf{P}}_{y,\theta}\left[\hat{\mathbf{P}}_{\xi_t,\Theta_t}\left(\tau_0^+ = +\infty\right)\mathbb{1}_{\{t<\tau_0^+\}}\right] \\ &= \hat{\mathbf{P}}_{y,\theta}\left(\tau_0^+ = +\infty\right) = \hat{H}^+_\theta(y).\end{aligned}$$

Using this and the Markov property of $((\xi,\Theta),\hat{\mathbf{P}})$ we prove that $\{\hat{H}^+_{\Theta_t}(\xi_t)1_{\{t<\tau_0^+\}}: t\geq 0\}$ is a $\hat{\mathbf{P}}$-martingale. □

Under the conditions of Proposition 4.1 we can define probability measures $\hat{\mathbf{P}}^{\downarrow}_{y,\theta}$ on the Skorokhod space $\mathbb{D}_{\mathbb{R}\times\mathcal{S}}$ by

$$\left.\frac{\mathrm{d}\hat{\mathbf{P}}^{\downarrow}_{y,\theta}}{\mathrm{d}\hat{\mathbf{P}}_{y,\theta}}\right|_{\mathcal{F}_t} := \frac{\hat{H}^+_{\Theta_t}(\xi_t)}{\hat{H}^+_{\theta}(y)}1_{\{t<\tau_0^+\}} \quad \forall y<0,\ \theta\in\mathcal{S},\ t\geq 0.$$

It follows by the theory of Doob's h-transform that for every $y<0$ and $\theta\in\mathcal{S}$ the process $((\xi,\Theta),\hat{\mathbf{P}}^{\downarrow}_{y,\theta})$ is a strong Markov process on the state space $(0,+\infty)\times\mathcal{S}$ with semigroup $(\hat{P}^{\downarrow}_t)_{t\geq 0}$ given by

$$\hat{P}^{\downarrow}_t f(z,\nu) = \frac{1}{\hat{H}^+_{\theta}(z)}\hat{\mathbf{P}}_{z,\nu}\left[\hat{H}^+_{\Theta_t}(\xi_t)f(\xi_t,\Theta_t)1_{\{t<\tau_0^+\}}\right] \quad \forall z<0, \nu\in\mathcal{S}, t\geq 0.$$

Since $\hat{H}^+_{\Theta_t}(\xi_t)1_{\{t<\tau_0^+\}}$ is a $\hat{\mathbf{P}}$-martingale, the semigroup $(\hat{P}^{\downarrow}_t)_{t\geq 0}$ is Markovian and accordingly the process $((\xi,\Theta),\hat{\mathbf{P}}^{\downarrow})$ has an infinite lifetime .

Proposition 4.2. *Suppose that (4.1) holds. For all $y<0$, $\theta\in\mathcal{S}$, $t\geq 0$ and $\Lambda\in\mathcal{F}_t$,*

$$\hat{\mathbf{P}}^{\downarrow}_{y,\theta}(\Lambda) = \lim_{q\to 0+}\hat{\mathbf{P}}_{y,\theta}\left(\Lambda, t<e_q \,|\, \tau_0^+>e_q\right).$$

Proof. Note that by the Markov property of $((\xi,\Theta),\hat{\mathbf{P}})$,

$$\begin{aligned}\hat{\mathbf{P}}_{y,\theta}\left(\Lambda;\ t<e_q<\tau_0^+\right) &= \int_t^{+\infty} q\mathrm{e}^{-qs}\hat{\mathbf{P}}_{y,\theta}(\Lambda;\ s<\tau_0^+)\mathrm{d}s \\ &= \int_0^{+\infty} q\mathrm{e}^{-q(s+t)}\hat{\mathbf{P}}_{y,\theta}(\Lambda;\ s+t<\tau_0^+)\mathrm{d}s \\ &= \mathrm{e}^{-qt}\hat{\mathbf{P}}_{y,\theta}\left(1_{\{\Lambda,t<\tau_0^+\}}\hat{\mathbf{P}}_{\xi_t,\Theta_t}(\tau_0^+>e_q)\right).\end{aligned}$$

Thus by the bounded convergence theorem,

$$\begin{aligned}\lim_{q\to 0+}\hat{\mathbf{P}}_{y,\theta}\left(\Lambda,\ t<e_q \,|\, \tau_0^+>e_q\right) &= \lim_{q\to 0+}\mathrm{e}^{-qt}\hat{\mathbf{P}}_{y,\theta}\left(1_{\{\Lambda,t<\tau_0^+\}}\frac{\hat{\mathbf{P}}_{\xi_t,\Theta_t}(\tau_0^+>e_q)}{\hat{\mathbf{P}}_{y,\theta}(\tau_0^+>e_q)}\right) \\ &= \hat{\mathbf{P}}_{y,\theta}\left(\frac{\hat{H}^+_{\Theta_t}(\xi_t)}{\hat{H}^+_{\theta}(y)}1_{\{\Lambda,t<\tau_0^+\}}\right) = \hat{\mathbf{P}}^{\downarrow}_{y,\theta}(\Lambda).\end{aligned}$$

□

The process $((\xi,\Theta),\hat{\mathbf{P}}^{\downarrow})$ is referred to as the *MAP conditioned to stay negative.*

PROPOSITION 4.3. *Suppose that (4.1) holds. For every $\theta \in \mathcal{S}$, there exists a probability measure $\hat{\mathbf{P}}^{\downarrow}_{0,\theta}$ on $\mathbb{D}_{\mathbb{R}\times\mathcal{S}}$ satisfying that $\xi_0 = 0$ and $\xi_t \neq 0$ for all $t > 0$, $\hat{\mathbf{P}}^{\downarrow}_{0,\theta}$-a.s., and that the process $(\xi_t, \Theta_t)_{t>0}$ under $\hat{\mathbf{P}}^{\downarrow}_{0,\theta}$ is a strong Markov process with the same transition rates as $((\xi, \Theta), \{\hat{\mathbf{P}}^{\downarrow}_{y,\theta} : y < 0, \theta \in \mathcal{S}\})$. Moreover we have*

$$\hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left[f(\xi_t, \Theta_t)\mathbb{1}_{\{t<\zeta\}}\right] = \frac{\hat{\mathrm{n}}^{+}_{\theta}\left[\hat{H}^{+}_{\nu_t}(-\epsilon_t)f(-\epsilon_t, \nu_t)\mathbb{1}_{\{t<\zeta\}}\right]}{\hat{\mathrm{n}}^{+}_{\theta}(\zeta = +\infty)} \tag{4.2}$$

for any $t > 0$ and nonnegative measurable function $f : \mathbb{R} \times \mathcal{S} \to \mathbb{R}^{+}$.

PROOF. To construct $\hat{\mathbf{P}}^{\downarrow}_{0,\theta}$ we use the transition kernels $\hat{\mathfrak{P}}^{\theta,x,u}$ defined in [23] (see also the arguments in Section 2.2.1). Recall that $\bar{R} = \inf\{t > 0 : t \in \bar{M}^{cl}\}$ and that under $\hat{\mathfrak{P}}^{\theta,x,u}$ the process $(\Theta_t, \xi_t, U_t)_{t\geq 0}$ starts from (θ, x, u) and $(\Theta_t, \xi_t, U_t)_{t>0}$ is a strong Markov process having the same transition rates as $((\Theta_t, \xi_t, U_t)_{t\geq 0}, \hat{\mathbf{P}}_{x,\theta})$. Note that $\hat{H}^{+}_{\theta}(y) = \lim_{q\to 0+} \hat{\mathbf{P}}_{y,\theta}\left(\tau^{+}_{0} > e_q\right)$ for $y < 0$ and $\theta \in \mathcal{S}$. It follows from the Markov property and the bouned convergence theorem that

$$\begin{aligned}
\hat{\mathfrak{P}}^{\theta,0,0}\left[\hat{H}^{+}_{\Theta_t}(\xi_t)\mathbb{1}_{\{t<\bar{R}\}}\right] &= \lim_{q\to 0+} \hat{\mathfrak{P}}^{\theta,0,0}\left[\hat{\mathbf{P}}_{\xi_t,\Theta_t}\left(\tau^{+}_{0} > e_q\right)\mathbb{1}_{\{t<\bar{R}\}}\right] \\
&= \lim_{q\to 0+} \mathrm{e}^{qt}\hat{\mathfrak{P}}^{\theta,0,0}\left(t < e_q < \bar{R}\right) \\
&= \lim_{q\to 0+} \hat{\mathrm{n}}^{+}_{\theta}\left(t < e_q < \zeta\right) \\
&= \hat{\mathrm{n}}^{+}_{\theta}(\zeta = +\infty).
\end{aligned}$$

Thus we can define a probability measure $\hat{\mathbf{P}}^{\downarrow}_{0,\theta}$ on $\mathbb{D}_{\mathbb{R}\times\mathcal{S}}$ by

$$\hat{\mathbf{P}}^{\downarrow}_{0,\theta}(A) := \frac{1}{\hat{\mathrm{n}}^{+}_{\theta}(\zeta = +\infty)}\hat{\mathfrak{P}}^{\theta,0,0}\left[\hat{H}^{+}_{\Theta_t}(\xi_t)\mathbb{1}_{\{t<\bar{R}\}}\mathbb{1}_A\right] \quad \forall A \in \mathcal{F}_t,\ t > 0. \tag{4.3}$$

One can easily show from the properties of $\hat{\mathfrak{P}}^{\theta,0,0}$ that under $\hat{\mathbf{P}}^{\downarrow}_{0,\theta}$ the process ξ_t leaves 0 instantaneously and never hits 0 again, and that the process $(\xi_t, \Theta_t)_{t\geq 0}$ is a Markov process whose transition rates satisfy

$$\hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left[\xi_{t+s} \in A, \Theta_{t+s} \in B \,|\, \xi_s, \Theta_s\right] = \hat{\mathbf{P}}^{\downarrow}_{\xi_s,\Theta_s}\left[\xi_t \in A, \Theta_t \in B\right]$$

for all $t, s \geq 0$, $A \in \mathcal{B}(\mathbb{R})$ and $B \in \mathcal{B}(\mathcal{S})$. Note that, by definition, under $\hat{\mathfrak{P}}^{\theta,0,0}$, U_t equals $-\xi_t$ for $t < \bar{R}$. Hence by (4.3) for every $t > 0$ and nonnegative measurable function $f : \mathbb{R} \times \mathcal{S} \to \mathbb{R}^{+}$, we have

$$\begin{aligned}
\hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left[f(\xi_t, \Theta_t)\mathbb{1}_{\{t<\zeta\}}\right] &= \frac{1}{\hat{\mathrm{n}}^{+}_{\theta}(\zeta = +\infty)}\hat{\mathfrak{P}}^{\theta,0,0}\left[\hat{H}^{+}_{\Theta_t}(-U_t)f(-U_t, \Theta_t)\mathbb{1}_{\{t<\bar{R}\}}\right] \\
&= \frac{1}{\hat{\mathrm{n}}^{+}_{\theta}(\zeta = +\infty)}\hat{\mathrm{n}}^{+}_{\theta}\left[\hat{H}^{+}_{\nu_t}(-\epsilon_t)f(-\epsilon_t, \nu_t)\mathbb{1}_{\{t<\zeta\}}\right].
\end{aligned}$$

In the second equality we use the fact that $\hat{\mathrm{n}}^{+}_{\theta}$ is the image measure of $(U_t, \Theta_t)_{t<\bar{R}}$ under $\hat{\mathfrak{P}}^{\theta,0,0}$. □

Remark 4.4. Suppose $\mathcal{S}=\{s_1,s_2,\cdots,s_n\}$ is a finite space and $((\xi,\Theta),\mathbf{P})$ is a MAP taking values in $\mathbb{R}\times\mathcal{S}$. For simplicity we assume the random variables $\Xi_{j,k}$ introduced in Example 3.1 are such that $\Xi_{j,k}\overset{d}{=}\Xi_{k,j}$ for all $1\le j,k\le n$. Suppose the process $(\Theta,\{\mathbf{P}_{x,\theta},\theta\in\mathcal{S}\})$ is irreducible and hence ergodic. Its invariant distribution is denoted by $\pi=(\pi_1,\pi_2,\cdots,\pi_n)$. In this case condition (WR) is satisfied by taking $\widetilde{\mathbf{P}}_{0,v}$ to be $\mathbf{P}_{0,v}$. Hence $\hat{\mathbf{P}}_{0,v}$ is the law of $(-\xi,\Theta)$ under $\mathbf{P}_{0,v}$. Let $\hat{\phi}_j(q):=\hat{\mathrm{n}}^+_j(1-\mathrm{e}^{-q\zeta})$ for $1\le j\le n$ and $q>0$. It is proved in [16] that

$$\lim_{q\to0+}\frac{\hat{\phi}_j(q)}{\hat{\phi}_k(q)}=\lim_{q\to0+}\frac{\hat{\mathrm{n}}^+_j(\zeta=+\infty)+\hat{\mathrm{n}}^+_j(1-\mathrm{e}^{-q\zeta},\zeta<+\infty)}{\hat{\mathrm{n}}^+_k(\zeta=+\infty)+\hat{\mathrm{n}}^+_k(1-\mathrm{e}^{-q\zeta},\zeta<+\infty)}=\frac{\pi_j}{\pi_k}.$$

It follows that if $\hat{\mathrm{n}}^+_j(\zeta=+\infty)>0$ for some (then for all) $1\le j\le n$, then there is a constant $c>0$ independent of j such that $\hat{\mathrm{n}}^+_j(\zeta=+\infty)=c\pi_j$. Since $\hat{\mathbf{P}}_{y,s_i}\left(\tau^+_0=+\infty\right)=\lim_{q\to0+}\hat{\mathbf{P}}_{0,s_i}\left(\bar{\xi}_{e_q}\le -y\right)$, we get by Proposition 2.3 and the bounded convergence theorem that

$$\hat{\mathbf{P}}_{y,s_i}\left(\tau^+_0=+\infty\right)=c\sum_{j=1}^n\hat{U}^+_{ij}(-y)\pi_j$$

where $\hat{U}^+_{ij}(-y)=\hat{\mathbf{P}}_{0,s_i}\left[\int_0^{\bar{L}_\infty}1_{\{\xi^+_t\le -y,\Theta^+_t=s_j\}}\mathrm{d}t\right]$. In [16], $\sum_{j=1}^n\hat{U}^+_{ij}(-y)\pi_j$ is used as the harmonic function to define a martingale change of measure under which the MAP is conditioned to stay negative.

Remark 4.5. Suppose $((\xi,\Theta),\mathbf{P})$ is a MAP where ξ is a (possibly killed) Lévy process on $\mathbb{R}$ whose law is independent of Θ and Θ has an invariant distribution. In this case condition (WR) is satisfied by taking $\widetilde{\mathbf{P}}_{0,v}=\mathbf{P}_{0,v}$ and hence $\hat{\mathbf{P}}_{0,v}$ is the law of $(-\xi,\Theta)$ under $\mathbf{P}_{0,v}$. We assume that for ξ, 0 is regular for both $(-\infty,0)$ and $(0,+\infty)$, in which case, both $((\xi,\Theta),\mathbf{P})$ and $((-\xi,\Theta),\mathbf{P})$ are upwards regular. We claim that (4.1) is satisfied if and only if the Lévy process ξ_t drifts to $+\infty$. To see the this, we first recall some known facts about Lévy processes. Let $\underline{L}_t$ be the local time of ξ at the running minima and n^- be the excursion measures at the minimum. In fact, n^- equals $\hat{\mathrm{n}}^+$ which is the excursion measure at the maximum of the dual process $-\xi$. Since 0 is regular for $(-\infty,0)$, there is a continuous version of $\underline{L}_t$ and a strictly positive constant l^- such that almost surely $\int_0^t\mathbb{1}_{\{\xi_s=\inf_{r\in[0,s]}\xi_r\}}\mathrm{d}s=l^-\,\underline{L}_t$ for all $t\ge0$. In this case, the inverse local time $\underline{L}^{-1}_t$ is a (killed) subordinator with Laplace exponent given by $\hat{\Phi}(q)=l^-q+\mathrm{n}^-(1-\mathrm{e}^{-q\zeta})$. It follows that $\underline{L}_\infty$ is exponentially distributed with parameter $\mathrm{n}^-(\zeta=+\infty)$. Hence $\mathrm{n}^-(\zeta=+\infty)>0$ if and only if ξ_t drifts to $+\infty$, in which case [11] showed further that $\mathrm{n}^+(\zeta)=l^++\mathrm{n}^+\left(1-\mathrm{e}^{-\zeta}\right)<+\infty$ where n^+ denotes the excursion measure at the maximum of ξ and l^+ is the drift coefficient for the inverse local time at the maximum.

5. Stationary overshoots and undershoots of MAP. Throughout this section we will assume that the modulator of $((\xi,\Theta),\mathbf{P})$

(5.1) Θ is positive recurrent with invariant distribution π which is fully supported on $\mathcal{S}$.

Definition 5.1. For $q>0$, let $\{T^{(q)}_n:n\ge0\}$ be a sequence of random variables such that $T^{(q)}_0=0$ and $\{T^{(q)}_{n+1}-T^{(q)}_n:n\ge0\}$ are independent and exponentially distributed random variables with mean $1/q$. Define

$$M^{(q),+}_n:=\Theta^+_{T^{(q)}_n}\quad\forall n\ge0.$$

We call $M^{(q),+} := \{M_n^{(q),+} : n \geq 0\}$ the q-embedded chain of the process $(\Theta_t^+)_{t\geq 0}$. Moreover, in the spirit of [27], we say that Θ^+ is a (nonarithmetic aperiodic) Harris recurrent process if Θ^+ has a (nonarithmetic aperiodic) Harris recurrent q-embedded chain for some $q > 0$.

Under the preceding assumption (5.1), together with the assumption that

$$\inf_{v\in\mathcal{S}} \left[\ell^+(v) + \mathrm{n}_v^+ \left(1 - \mathrm{e}^{-\zeta}\right)\right] > 0 \text{ and } \mathrm{n}_v^+(\zeta) < +\infty \text{ for every } v \in \mathcal{S}, \tag{5.2}$$

it follows by Corollary 2.15 that

$$\pi^+(\cdot) = \frac{1}{\mathbf{P}_{0,\pi}\left[\bar{L}_1\right]}\mathbf{P}_{0,\pi}\left[\int_0^1 \mathbb{1}_{\{\Theta_s\in\cdot\}}\mathrm{d}\bar{L}_s\right] \tag{5.3}$$

is an invariant distribution for Θ^+ and hence for $M^{(q),+}$. It follows by [24, Theorem (5.1)] that

$$\pi(\mathrm{d}v) = \frac{1}{c_{\pi^+}}\left[\ell^+(v)\pi^+(\mathrm{d}v) + \int_{\mathcal{S}} \mathrm{n}_\theta^+\left(\int_0^\zeta \mathbb{1}_{\{\nu_t\in\mathrm{d}v\}}\mathrm{d}t\right)\pi^+(\mathrm{d}\theta)\right]$$

where $c_{\pi^+} := \int_{\mathcal{S}}\left[\ell^+(\theta) + \mathrm{n}_\theta^+(\zeta)\right]\pi^+(\mathrm{d}\theta)$ is a positive constant.

LEMMA 5.1. *Assume that (5.1) and (5.2) hold and, further, that $\mathbf{P}_{0,\pi^+}\left[\xi_1^+\right] < +\infty$ where π^+ given in (5.3) is fully supported on $\mathcal{S}$. Suppose that the continuous part of ξ^+ can be represented by $\int_0^t a^+(\Theta_s^+)\mathrm{d}s$ for some strictly positive measurable function a^+ on $\mathcal{S}$. Then for all $q > 0$, we have*

$$\mu^+ := \int_{\mathcal{S}} a^+(\phi)\pi^+(\mathrm{d}\phi) + \int_{\mathcal{S}\times\mathbb{R}^+} \bar{\Pi}_\phi^+(y)\pi^+(\mathrm{d}\phi)\mathrm{d}y = q\mathbf{P}_{0,\pi^+}\left[\xi_{e_q}^+\right] < +\infty,$$

where $\bar{\Pi}_\phi^+(y) := \Pi^+(\phi, \mathcal{S}, (y, +\infty))$.

PROOF. Using that $\mathbf{P}_{0,\pi^+}\left[\xi_1^+\right] < +\infty$ and the subadditivity of $t \mapsto \mathbf{P}_{0,\pi^+}\left[\xi_t^+\right]$, one can show in the same way as in the proof of Lemma 2.4 that $\mathbf{P}_{0,\pi^+}\left[\xi_t^+\right] < +\infty$ for all $t > 0$ and $\mathbf{P}_{0,\pi^+}\left[\xi_{e_q}^+\right] < +\infty$ for all $q > 0$. We note that for every $t > 0$,

$$\xi_t^+ = \int_0^t a^+(\Theta_s^+)\mathrm{d}s + \sum_{0\leq s\leq t} \Delta\xi_s^+ \mathbb{1}_{\{\Delta\xi_s^+>0\}},$$

where $\Delta\xi_s^+ = \xi_s^+ - \xi_{s-}^+$. By Proposition 2.5 and Fubini's theorem, we have

$$\begin{aligned}\mathbf{P}_{0,\theta}\left[\sum_{0\leq s\leq t}\Delta\xi_s^+\mathbb{1}_{\{\Delta\xi_s^+>0\}}\right] &= \mathbf{P}_{0,\theta}\left[\int_0^t \mathrm{d}s\int_{\mathcal{S}\times\mathbb{R}^+} y\Pi^+(\Theta_s^+, \mathrm{d}\phi, \mathrm{d}y)\right] \\ &= \mathbf{P}_{0,\theta}\left[\int_0^t \mathrm{d}s\int_0^{+\infty} \bar{\Pi}_{\Theta_s^+}^+(z)\mathrm{d}z\right] \qquad (5.4)\end{aligned}$$

for every $\theta \in \mathcal{S}$. Hence

$$\begin{aligned}\mathbf{P}_{0,\pi^+}[\xi_t^+] &= \mathbf{P}_{0,\pi^+}\left[\int_0^t \mathrm{d}s \left(a^+(\Theta_s^+) + \int_0^{+\infty} \bar{\Pi}^+_{\Theta_s^+}(z)\mathrm{d}z\right)\right] \\ &= t\left(\int_{\mathcal{S}} a^+(\phi)\pi^+(\mathrm{d}\phi) + \int_{\mathcal{S}\times\mathbb{R}^+} \bar{\Pi}^+_\phi(z)\pi^+(\mathrm{d}\phi)\mathrm{d}z\right) \\ &= t\mu^+.\end{aligned}$$

In the second equality we use the fact that π^+ is an invariant distribution for Θ^+. Consequently we have

$$\mathbf{P}_{0,\pi^+}\left[\xi^+_{e_q}\right] = q\int_0^{+\infty} \mathrm{e}^{-qt}\mathbf{P}_{0,\pi^+}\left[\xi_t^+\right]\mathrm{d}t = \frac{\mu^+}{q}.$$

□

Under the assumptions of Lemma 5.1, the measure $\rho^{\ominus}$ given below is a probability measure on $\mathbb{R}^+ \times \mathcal{S}$,

$$\rho^{\ominus}(\mathrm{d}z,\mathrm{d}v) := \frac{1}{\mu^+}\left[a^+(v)\pi^+(\mathrm{d}v)\delta_0(\mathrm{d}z) + 1_{\{z>0\}}\int_{\mathcal{S}\times\mathbb{R}^+} \pi^+(\mathrm{d}\phi)\mathrm{d}y\, \Pi^+(\phi,\mathrm{d}v,\mathrm{d}z+y)\right]. \tag{5.5}$$

We will show in the following that $\rho^{\ominus}$ is the stationary distribution for the overshoots of the MAP, assuming additionally that,

(5.6) the ssMp underlying $((\xi,\Theta),\mathbf{P})$ via Lamperti-Kiu transform is a Feller process,

and the modulator

(5.7) Θ^+ of $((\xi^+,\Theta^+),\mathbf{P})$ is a nonarithmetic aperiodic Harris recurrent process.

The key of the proof is the application of Markov renewal theory developed in [2]. Suppose that $\{M_n^{(q),+} = \Theta^+_{T_n^{(q)}} : n \geq 0\}$ is a nonarithmetic aperiodic Harris recurrent q-embedded chain of $((\xi^+,\Theta^+),\mathbf{P})$. Define

$$S_n^{(q),+} := \xi^+_{T_n^{(q)}}, \quad N_n^{(q),+} := \bar{L}^{-1}_{T_n^{(q)}} \quad \forall n \geq 0.$$

One can easily show that $(M_n^{(q),+}, S_n^{(q),+})_{n\geq 0}$ and $(M_n^{(q),+}, N_n^{(q),+})_{n\geq 0}$ both are Markov renewal processes in the sense of [2]. We shall first consider the process $(M_n^{(q),+}, S_n^{(q),+})_{n\geq 0}$. For every $\theta \in \mathcal{S}$, let

$$F_\theta(\mathrm{d}v,\mathrm{d}z) := \mathbf{P}_{0,\theta}(M_1^{(q),+} \in \mathrm{d}v, S_1^{(q),+} \in \mathrm{d}z) = \int_0^{+\infty} q\mathrm{e}^{-qt}\mathbf{P}_{0,\theta}\left(\Theta_t^+ \in \mathrm{d}v,\ \xi_t^+ \in \mathrm{d}z\right)\mathrm{d}t. \tag{5.8}$$

Let $F_\theta^{*0}(\mathrm{d}v,\mathrm{d}z) := \delta_\theta(\mathrm{d}v)\delta_0(\mathrm{d}z)$ and F_θ^{*n} be the n-th convolution of F_θ for $n \geq 1$. Then $\sum_{n=0}^{+\infty} F_\theta^{*n}(\mathrm{d}v,\mathrm{d}z)$ is the renewal measure of Markov renewal process $(M_n^{(q),+}, S_n^{(q),+})_{n\geq 0}$. Note that $\mathbf{P}_{0,\pi^+}\left[S_1^{(q),+}\right] = \mathbf{P}_{0,\pi^+}\left[\xi^+_{e_q}\right] = \mu^+/q$. Given (5.7), it follows by [2, Theorem 2.1] that

$$\lim_{y\to+\infty}\int_{\mathcal{S}\times[0,y]} g(v,y-z)\sum_{n=0}^{+\infty} F_\theta^{*n}(\mathrm{d}v,\mathrm{d}z) = \frac{q}{\mu^+}\int_{\mathcal{S}\times\mathbb{R}^+} g(v,z)\pi^+(\mathrm{d}v)\mathrm{d}z \tag{5.9}$$

for every $\theta \in \mathcal{S}$ and every measurable function $g : \mathcal{S} \times \mathbb{R}^+ \to \mathbb{R}$ satisfying the following two conditions:

(i) for each $v \in \mathcal{S}$, the set of discontinuous points of $z \mapsto g(v, z)$ has zero Lebesgue measure;
(ii) $\int_{\mathcal{S}} \sum_{n=0}^{+\infty} \sup_{z \in [np,(n+1)p)} |g(v, z)| \pi^+(\mathrm{d}v) < +\infty$ for some $p > 0$.

We use $\mathcal{M}$ to denote the space of measurable functions satisfying both of the above conditions. In view of the fact that $\mathbf{P}_{0,\theta}\left(T_n^{(q)} \in \mathrm{d}t\right) = \frac{q^n t^{n-1}}{(n-1)!}\mathrm{e}^{-qt}\mathrm{d}t$ for $n \geq 1$, we have

$$\begin{aligned} U_\theta^+(\mathrm{d}v, \mathrm{d}z) &= \int_0^{+\infty} \mathbf{P}_{0,\theta}\left(\Theta_t^+ \in \mathrm{d}v,\ \xi_t^+ \in \mathrm{d}z\right)\mathrm{d}t \\ &= \sum_{n=1}^{+\infty} \int_0^{+\infty} \mathrm{e}^{-qt}\frac{(qt)^{n-1}}{(n-1)!}\mathbf{P}_{0,\theta}\left(\Theta_t^+ \in \mathrm{d}v,\ \xi_t^+ \in \mathrm{d}z\right)\mathrm{d}t \\ &= \frac{1}{q}\sum_{n=1}^{+\infty} \mathbf{P}_{0,\theta}\left(M_n^{(q),+} \in \mathrm{d}v,\ S_n^{(q),+} \in \mathrm{d}z\right) \\ &= \frac{1}{q}\left[\sum_{n=0}^{+\infty} F_\theta^{*n}(\mathrm{d}v, \mathrm{d}z) - \delta_\theta(\mathrm{d}v)\delta_0(\mathrm{d}z)\right]. \end{aligned}$$

This and (5.9) imply that for every $\theta \in \mathcal{S}$ and every $g \in \mathcal{M}$,

$$\lim_{y\to+\infty} \int_{\mathcal{S}\times[0,y]} g(v, y - z)U_\theta^+(\mathrm{d}v, \mathrm{d}z) = \frac{1}{\mu^+}\int_{\mathcal{S}\times\mathbb{R}^+} g(v, z)\pi^+(\mathrm{d}v)\mathrm{d}z. \tag{5.10}$$

Remark 5.2. It is easy to see that $g \in \mathcal{M}$ if, in particular, $z \mapsto g(v, z)$ is right continuous on $[0, +\infty)$ and there is a measurable function $f : \mathcal{S} \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $|g(v, z)| \leq f(v, z)$ for all $(v, z) \in \mathcal{S} \times \mathbb{R}^+$, $z \mapsto f(v, z)$ is a monotone function on $\mathbb{R}^+$ and $\int_{\mathcal{S}\times\mathbb{R}^+} f(v, z)\pi^+(\mathrm{d}v)\mathrm{d}z < +\infty$. In fact this sufficient condition for $g \in \mathcal{M}$ is easy to be verified and will be used later in our proofs where the Markov renewal theory is applied.

Proposition 5.3. *Suppose (5.6), (5.7) and the conditions in Lemma 5.1 hold. For every $\theta \in \mathcal{S}$, the joint probability measures on $\mathcal{S} \times \mathbb{R}^- \times \mathcal{S} \times \mathbb{R}^+$*

$$\mathbf{P}_{0,\theta}\left(\Theta_{\tau_x^+-} \in \mathrm{d}v, \xi_{\tau_x^+-} - x \in \mathrm{d}y, \Theta_{\tau_x^+} \in \mathrm{d}\phi, \xi_{\tau_x^+} - x \in \mathrm{d}z\right)$$

converges weakly to a probability measure ρ given by

$$\begin{aligned} \rho(\mathrm{d}v, \mathrm{d}y, \mathrm{d}\phi, \mathrm{d}z) \;:=\; & \frac{1}{\mu^+}\Big[\mathbb{1}_{\{y<0,z>0\}}\ell^+(v)\Pi(v, \mathrm{d}\phi, \mathrm{d}z - y)\pi^+(\mathrm{d}v)\mathrm{d}y \\ & +\mathbb{1}_{\{y<0,z>0\}}\mathrm{d}y\int_{\mathcal{S}}\pi^+(\mathrm{d}\varphi)\mathrm{n}_\varphi^+\left(\int_0^\zeta \mathbb{1}_{\{\epsilon_s\leq -y,\nu_s\in\mathrm{d}v\}}\Pi(v, \mathrm{d}\phi, \mathrm{d}z - y)\mathrm{d}s\right) \\ & +a^+(v)\pi^+(\mathrm{d}v)\delta_0(\mathrm{d}y)\delta_0(\mathrm{d}z)\delta_v(\mathrm{d}\phi)\Big] \end{aligned}$$

as $x \to +\infty$. In particular, $\mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+} - x \in \mathrm{d}z,\ \Theta_{\tau_x^+} \in \mathrm{d}\phi\right)$ *converges weakly to* $\rho^{\ominus}(\mathrm{d}z, \mathrm{d}\phi)$ *given by (5.5), and* $\mathbf{P}_{0,\theta}\left(\xi_{\tau_x^+-} - x \in \mathrm{d}y,\ \Theta_{\tau_x^+-} \in \mathrm{d}v\right)$ *converges weakly to a probability measure* $\rho^{\oplus}(\mathrm{d}y, \mathrm{d}v)$ *given by*

$$\begin{aligned}\rho^{\oplus}(\mathrm{d}y, \mathrm{d}v) \;&:=\; \frac{1}{\mu^+}\big[a^+(v)\pi^+(\mathrm{d}v)\delta_0(\mathrm{d}y) + \mathbb{1}_{\{y<0\}}\ell^+(v)\bar{\Pi}_v(-y)\pi^+(\mathrm{d}v)\mathrm{d}y \\ &\quad + \mathbb{1}_{\{y<0\}}\mathrm{d}y\int_{\mathcal{S}}\pi^+(\mathrm{d}\phi)\mathrm{n}^+_{\phi}\left(\int_0^{\zeta}\bar{\Pi}_v(-y)\mathbb{1}_{\{\epsilon_r\le -y, \nu_r\in \mathrm{d}v\}}\mathrm{d}r\right)\big].\end{aligned}$$

Here $\bar{\Pi}_v(-y) := \Pi(v, \mathcal{S}, (-y, +\infty))$.

Proof. First we claim that ρ given above is a probability measure. Integrating $\rho(\mathrm{d}v, \mathrm{d}y, \mathrm{d}\phi, \mathrm{d}z)$ over the variables v and y, we get that

$$\begin{aligned}&\frac{1}{\mu^+}\Bigg[a^+(\phi)\pi^+(\mathrm{d}\phi)\delta_0(\mathrm{d}z) + \mathbb{1}_{\{z>0\}}\int_{\mathcal{S}\times\mathbb{R}^+}\pi^+(\mathrm{d}v)\mathrm{d}y\,\ell^+(v)\Pi(v, \mathrm{d}\phi, \mathrm{d}z + y) \\ &\quad + \mathbb{1}_{\{z>0\}}\int_{\mathcal{S}}\pi^+(d\varphi)\mathrm{n}^+_{\varphi}\left(\int_0^{\zeta}\mathrm{d}s\int_{\epsilon_s}^{+\infty}\mathrm{d}y\,\mathbb{1}_{\{\nu_s\in\mathrm{d}v\}}\Pi(\nu_s, \mathrm{d}\phi, \mathrm{d}z + y)\right)\Bigg] \\ &\quad = \frac{1}{\mu^+}\Bigg[a^+(\phi)\pi^+(\mathrm{d}\phi)\delta_0(\mathrm{d}z) + \mathbb{1}_{\{z>0\}}\int_{\mathcal{S}\times\mathbb{R}^+}\pi^+(\mathrm{d}v)\mathrm{d}y\,\ell^+(v)\Pi(v, \mathrm{d}\phi, \mathrm{d}z + y) \\ &\quad + \mathbb{1}_{\{z>0\}}\int_{\mathcal{S}\times\mathbb{R}^+}\pi^+(d\varphi)\mathrm{d}u\,\mathrm{n}^+_{\varphi}\left(\int_0^{\zeta}\mathbb{1}_{\{\nu_s\in\mathrm{d}v\}}\Pi(\nu_s, \mathrm{d}\phi, \mathrm{d}z + \epsilon_s + u)\right)\Bigg] \\ &\quad = \frac{1}{\mu^+}\Bigg[a^+(\phi)\pi^+(\mathrm{d}\phi)\delta_0(\mathrm{d}z) + \mathbb{1}_{\{z>0\}}\int_{\mathcal{S}\times\mathbb{R}^+}\pi^+(\mathrm{d}v)\mathrm{d}y\Pi^+(v, \mathrm{d}\phi, \mathrm{d}z + y)\Bigg] \\ &\quad = \rho^{\ominus}(\mathrm{d}z, \mathrm{d}\phi).\end{aligned}$$

The first equality follows from a change of variable and Fubini's theorem, and the second equality follows from Proposition 2.5. This implies that ρ is a probability measure and $\rho^{\ominus}$ is its marginal law. Similarly, by integrating $\rho(\mathrm{d}v, \mathrm{d}y, \mathrm{d}\phi, \mathrm{d}z)$ over the variables ϕ and z, we can show that $\rho^{\oplus}$ is also a marginal law of ρ. Next we prove the weak convergence. Suppose $f, g : \mathcal{S}\times\mathbb{R}\to\mathbb{R}$ are bounded continuous functions. It follows by Proposition 2.7 that for any $x > 0$,

$$\begin{aligned}&\mathbf{P}_{0,\theta}\left[f(\Theta_{\tau_x^+-}, \xi_{\tau_x^+-} - x)g(\Theta_{\tau_x^+}, \xi_{\tau_x^+} - x)\mathbb{1}_{\{\xi_{\tau_x^+}>x\}}\right] \\ &\quad = \int_{\mathcal{S}\times[0,x]}U^+_{\theta}(\mathrm{d}v, \mathrm{d}z)\,\big[\ell^+(v)f(v, z - x)G(v, x - z) \\ &\quad + \mathrm{n}^+_v\left(\int_0^{\zeta}f(\nu_s, z - x - \epsilon_s)G(\nu_s, x - z + \epsilon_s)\mathrm{d}s\right)\big]. \qquad (5.11)\end{aligned}$$

where $G(v, u) = \int_{\mathcal{S}\times(u,+\infty)}g(\phi, y - u)\Pi(v, \mathrm{d}\phi, \mathrm{d}y)$. One can easily show that the condition given in Remark 5.2 is satisfied by the function

$$(v, z) \mapsto \ell^+(v)f(v, -z)G(v, z) + \mathrm{n}^+_v\left(\int_0^{\zeta}f(\nu_s, -z - \epsilon_s)G(\nu_s, z + \epsilon_s)\mathrm{d}s\right).$$

Hence by (5.10), the integral in the right-hand side converges to

$$\frac{1}{\mu^+}\int_{\mathcal{S}\times\mathbb{R}^+}\pi^+(\mathrm{d}v)\mathrm{d}z\left[\ell^+(v)f(v,-z)G(v,z)+\mathrm{n}^+_v\left(\int_0^\zeta f(\nu_s,-z-\epsilon_s)G(\nu_s,z+\epsilon_s)\mathrm{d}s\right)\right]. \tag{5.12}$$

By Fubini's theorem, we have

$$\begin{aligned}\int_{\mathcal{S}\times\mathbb{R}^+}&\pi^+(\mathrm{d}v)\mathrm{d}z\mathrm{n}^+_v\left(\int_0^\zeta f(\nu_s,-z-\epsilon_s)G(\nu_s,z+\epsilon_s)\mathrm{d}s\right)\\ &=\int_{\mathcal{S}}\pi^+(\mathrm{d}v)\mathrm{n}^+_v\left(\int_0^{+\infty}\int_0^\zeta f(\nu_s,-z-\epsilon_s)G(\nu_s,z+\epsilon_s)\mathrm{d}s\mathrm{d}z\right)\\ &=\int_{\mathcal{S}}\pi^+(\mathrm{d}v)\mathrm{n}^+_v\left(\int_0^\zeta\mathrm{d}s\int_{\epsilon_s}^{+\infty}f(\nu_s,-y)G(\nu_s,y)\mathrm{d}y\right)\\ &=\int_{\mathcal{S}\times\mathbb{R}^+}\pi^+(\mathrm{d}v)\mathrm{d}y\mathrm{n}^+_v\left(\int_0^\zeta 1_{\{\epsilon_s\le y\}}f(\nu_s,-y)G(\nu_s,y)\mathrm{d}s\right).\end{aligned} \tag{5.13}$$

Next we deal with the creeping event $\{\xi_{\tau_x^+}=x\}$. Note that

$$\begin{aligned}F_\theta(\mathrm{d}v,\mathrm{d}z)&=\int_0^{+\infty}q\mathrm{e}^{-qt}\mathbf{P}_{0,\theta}\left(\Theta^+_t\in\mathrm{d}v,\xi^+_t\in\mathrm{d}z\right)\mathrm{d}t\\ &=q\int_0^{+\infty}\mathbf{P}_{0,\theta}\left(\Theta^+_t\in\mathrm{d}v,\xi^+_t\in\mathrm{d}z,t<e_q\right)\mathrm{d}t.\end{aligned}$$

This equation implies that $F_\theta(\mathrm{d}v,\mathrm{d}z)/q$ can be viewed as the potential measure of the non-decreasing MAP (ξ^+,Θ^+) killed by an independent exponential time e_q. In fact, we can verify that this killed process is still a nondecreasing MAP and satisfies all the conditions in Lemma 2.1. Hence by Lemma 2.1 $F_\theta(\mathrm{d}v,\mathrm{d}z)$ has a density function $f_\theta(\mathrm{d}v,z)$ with respect to the Lebesgue measure $\mathrm{d}z$ such that

$$\mathbf{P}_{0,\theta}\left(h(\Theta^+_{T^+_x});\xi^+_{T^+_x}=x,T^+_x<e_q\right)=\frac{1}{q}\int_{\mathcal{S}}a^+(v)h(v)f_\theta(\mathrm{d}v,x) \tag{5.14}$$

for every nonnegative measurable function $h:\mathcal{S}\to\mathbb{R}$ and almost every $x>0$. We claim that $x\mapsto\mathbf{P}_{0,\theta}\left(h(\Theta^+_{T^+_x});\xi^+_{T^+_x}=x,T^+_x<e_q\right)=\mathbf{P}_{0,\theta}\left(\mathrm{e}^{-qT^+_x}h(\Theta^+_{T^+_x});\xi^+_{T^+_x}=x\right)$ is right continuous on $[0,+\infty)$ if in particular h is a bounded continuous function. To see this, we take an arbitrary sequence $x_n,x\in\mathbb{R}^+$ and $x_n\downarrow x$. Since $\xi^+_{T^+_x}=\xi_{\tau^+_x}$ we have

$$\begin{aligned}&\left|\mathbf{P}_{0,\theta}\left(\mathrm{e}^{-qT^+_{x_n}}h(\Theta^+_{T^+_{x_n}});\xi^+_{T^+_{x_n}}=x_n\right)-\mathbf{P}_{0,\theta}\left(\mathrm{e}^{-qT^+_x}h(\Theta^+_{T^+_x});\xi^+_{T^+_x}=x\right)\right|\\ &\le\left|\mathbf{P}_{0,\theta}\left[\mathrm{e}^{-qT^+_{x_n}}h(\Theta^+_{T^+_{x_n}})\left(1_{\{\xi^+_{T^+_{x_n}}=x_n\}}-1_{\{\xi^+_{T^+_x}=x\}}\right)\right]\right|\\ &+\left|\mathbf{P}_{0,\theta}\left[\mathrm{e}^{-qT^+_{x_n}}h(\Theta^+_{T^+_{x_n}})-\mathrm{e}^{-qT^+_x}h(\Theta^+_{T^+_x});\xi^+_{T^+_x}=x\right]\right|\\ &\le\|h\|_\infty\mathbf{P}_{0,\theta}\left(\{\xi_{\tau^+_{x_n}}=x_n\}\Delta\{\xi_{\tau^+_x}=x\}\right)+\mathbf{P}_{0,\theta}\left[\left|\mathrm{e}^{-qT^+_{x_n}}h(\Theta^+_{T^+_{x_n}})-\mathrm{e}^{-qT^+_x}h(\Theta^+_{T^+_x})\right|\right].\end{aligned}$$

In view of (2.22) and the fact that $T^+_{x_n} \downarrow T^+_x$ and $\Theta^+_{T^+_{x_n}} \to \Theta^+_{T^+_x}$ $\mathbf{P}_{0,\theta}$-a.s., we get by the above inequality and the bounded convergence theorem that

$$\lim_{n\to+\infty} \mathbf{P}_{0,\theta}\left(\mathrm{e}^{-qT^+_{x_n}} h(\Theta^+_{T^+_{x_n}}); \xi^+_{T^+_{x_n}} = x_n\right) = \mathbf{P}_{0,\theta}\left(\mathrm{e}^{-qT^+_x} h(\Theta^+_{T^+_x}); \xi^+_{T^+_x} = x\right).$$

Hence we prove the claim. Now we set $f_\theta(\mathrm{d}v, x) = \frac{q}{a^+(v)}\mathbf{P}_{0,\theta}\left(\Theta^+_{T^+_x} \in \mathrm{d}v, \xi^+_{T^+_x} = x, T^+_x < e_q\right)$ for every $x > 0$. The above arguments shows that $x \mapsto a^+(v) f_\theta(\mathrm{d}v, x)$ is right continuous on $(0, +\infty)$ in the sense of vague convergence and (5.14) holds for every $x > 0$ and every nonnegative measurable function $h : \mathcal{S} \to \mathbb{R}$. Since

$$\begin{aligned} U^+_\theta(\mathrm{d}v, \mathrm{d}z) &= \frac{1}{q}\sum_{n=0}^{+\infty} F^{*(n+1)}_\theta(\mathrm{d}v, \mathrm{d}z) \\ &= \frac{1}{q}\int_{\mathcal{S}\times[0,z]} F_\phi(\mathrm{d}v, \mathrm{d}z - y)\sum_{n=0}^{+\infty} F^{*n}_\theta(\mathrm{d}\phi, \mathrm{d}y), \end{aligned}$$

we can take the density function $u^+_\theta(\mathrm{d}v, z)$ of $U^+_\theta(\mathrm{d}v, \mathrm{d}z)$ to be such that

$$u^+_\theta(\mathrm{d}v, z) = \frac{1}{q}\int_{\mathcal{S}\times[0,z]} f_\phi(\mathrm{d}v, z - y)\sum_{n=0}^{+\infty} F^{*n}_\theta(\mathrm{d}\phi, \mathrm{d}y) \quad \forall z > 0. \tag{5.15}$$

For $n \geq 1$,

$$\begin{aligned} F^{*n}_\theta(\mathrm{d}v, \mathrm{d}z) &= \int_0^{+\infty} \mathbf{P}_{0,\theta}\left(\Theta^+_{T^{(q)}_n} \in \mathrm{d}v, \xi^+_{T^{(q)}_n} \in \mathrm{d}z\right)\mathrm{d}t \\ &= \int_0^{+\infty} \frac{q^n t^{n-1}}{(n-1)!}\mathrm{e}^{-qt}\mathbf{P}_\theta(\Theta^+_t \in \mathrm{d}v, \xi^+_t \in \mathrm{d}z)\mathrm{d}t, \end{aligned}$$

Obviously $F^{*n}_\theta(\mathrm{d}v, \mathrm{d}z)$ is absolutely continuous with respect to $U^+_\theta(\mathrm{d}v, \mathrm{d}z)$, and hence $F^{*n}_\theta(\mathrm{d}v, \mathrm{d}z)$ has a density function with respect to the Lebesgue measure $\mathrm{d}z$ which is denoted by $f^{*n}_\theta(\mathrm{d}v, z)$. In view of this, $u^+_\theta(\mathrm{d}v, z)$ given in (5.15) can be represented by

$$u^+_\theta(\mathrm{d}v, z) = \frac{1}{q} f_\theta(\mathrm{d}v, z) + \frac{1}{q}\int_0^z \mathrm{d}y \int_{\mathcal{S}} f_\phi(\mathrm{d}v, z - y)\sum_{n=1}^{+\infty} f^{*n}_\theta(\mathrm{d}\phi, y).$$

Using this expression and the fact that $z \mapsto a^+(v) f_\phi(\mathrm{d}v, z)$ is right continuous on $(0, +\infty)$, we can show that $x \mapsto a^+(v) u^+_\theta(\mathrm{d}v, x)$ is right continuous on $(0, +\infty)$ in the sense of vague convergence. Hence $u^+_\theta(\mathrm{d}v, z)$ given in (5.15) is the density function taken in Proposition 2.9, and we have

$$\begin{aligned} &\mathbf{P}_{0,\theta}\left[f(\Theta_{\tau^+_x-}, \xi_{\tau^+_x-} - x) g(\Theta_{\tau^+_x}, \xi_{\tau^+_x} - x)\mathbb{1}_{\{\xi_{\tau^+_x} = x\}}\right] \\ &= \int_{\mathcal{S}} a^+(v) f(v, 0) g(v, 0) u^+_\theta(\mathrm{d}v, x) \\ &= \frac{1}{q}\int_{\mathcal{S}\times[0,x]} \sum_{n=0}^{+\infty} F^{*n}_\theta(\mathrm{d}\phi, \mathrm{d}y)\int_{\mathcal{S}} f(v, 0) g(v, 0) a^+(v) f_\phi(\mathrm{d}v, x - y) \end{aligned}$$

for every $x > 0$. Again by Remark 5.2 we can show that $(\phi, z) \mapsto \int_{\mathcal{S}} f(v,0)g(v,0)a^+(v)f_\phi(\mathrm{d}v, z) = q\mathbf{P}_{0,\phi}\left[f(\Theta^+_{T_z^+},0)g(\Theta^+_{T_z^+},0); T_z^+ < e_q\right] \in \mathcal{M}$. Hence by (5.9) the integral in the right-hand side converges, as $x \to +\infty$, towards

$$
\begin{aligned}
&\frac{1}{\mu^+}\int_{\mathcal{S}\times\mathbb{R}^+} \pi^+(\mathrm{d}\phi)\mathrm{d}y \int_{\mathcal{S}} a^+(v)f(v,0)g(v,0)f_\phi(\mathrm{d}v, y) \\
&\quad = \frac{1}{\mu^+}\int_{\mathcal{S}} \pi^+(\mathrm{d}\phi)\int_{\mathcal{S}\times\mathbb{R}^+} a^+(v)f(v,0)g(v,0)F_\phi(\mathrm{d}v,\mathrm{d}y) \\
&\quad = \frac{1}{\mu^+}\int_{\mathcal{S}} \pi^+(\mathrm{d}\phi)\mathbf{P}_{0,\phi}\left[a^+(M_1^{(q),+})f(M_1^{(q),+},0)g(M_1^{(q),+},0)\right] \\
&\quad = \frac{1}{\mu^+}\int_{\mathcal{S}} a^+(v)f(v,0)g(v,0)\pi^+(\mathrm{d}v). \qquad (5.16)
\end{aligned}
$$

In the final equality we use the fact that π^+ is an invariant distribution for $(M_n^{(q),+})_{n\geq 0}$. Combining (5.12), (5.13) and (5.16) we get

$$
\begin{aligned}
&\mathbf{P}_{0,\theta}\left[f(\Theta_{\tau_x^+-}, \xi_{\tau_x^+-} - x)g(\Theta_{\tau_x^+}, \xi_{\tau_x^+} - x)\right] \\
&\quad \to \frac{1}{\mu^+}\left[\int_{\mathcal{S}\times\mathbb{R}^+} \pi^+(\mathrm{d}v)\mathrm{d}z\ell^+(v)f(v,-z)G(v,z) + \int_{\mathcal{S}} \pi^+(\mathrm{d}v)a^+(v)f(v,0)g(v,0)\right. \\
&\quad \left. + \int_{\mathcal{S}\times\mathbb{R}^+} \pi^+(\mathrm{d}v)\mathrm{d}y\mathrm{n}_v^+\left(\int_0^\zeta \mathbb{1}_{\{\epsilon_s\leq y\}} f(\nu_s, -y)G(\nu_s, y)\mathrm{d}s\right)\right] \qquad \text{as } x \to +\infty,
\end{aligned}
$$

which yields the first assertion of this proposition. The second and third assertion follow immediately from the above equation by setting $f \equiv 1$ and $g \equiv 1$ respectively. □

In the remaining of this section we consider the nondecreasing MAP $(\bar{L}^{-1}, \Theta^+)$. The ordinate $\bar{L}^{-1}$ can be represented by

$$
\bar{L}_t^{-1} = \int_0^t \ell^+(\Theta_s^+)\mathrm{d}s + \sum_{s\leq t} \Delta \bar{L}_s^{-1} \quad \forall t \geq 0 \qquad (5.17)
$$

where $\Delta\bar{L}_s^{-1} = \bar{L}_s^{-1} - \bar{L}_{s-}^{-1}$. Note that for any $t \geq 0$, assuming (5.1) and (5.2),

$$
\begin{aligned}
\mathbf{P}_{0,\pi^+}\left[\bar{L}_t^{-1}\right] &= \mathbf{P}_{0,\pi^+}\left[\int_0^t \ell^+(\Theta_s^+)\mathrm{d}s + \sum_{s\leq t}\Delta\bar{L}_s^{-1}\right] \\
&= \mathbf{P}_{0,\pi^+}\left[\int_0^t \left(\ell^+(\Theta_s^+) + \mathrm{n}^+_{\Theta_s^+}(\zeta)\right)\mathrm{d}s\right] \\
&= \int_0^t \mathbf{P}_{0,\pi^+}\left[\left(\ell^+(\Theta_s^+) + \mathrm{n}^+_{\Theta_s^+}(\zeta)\right)\right]\mathrm{d}s \\
&= t\int_{\mathcal{S}} \left(\ell^+(\theta) + \mathrm{n}^+_\theta(\zeta)\right)\pi^+(\mathrm{d}\theta) = tc_{\pi^+}.
\end{aligned}
$$

In the last equality we use the fact that π^+ is an invariant distribution for $(\Theta^+_t)_{t\geq 0}$. If we consider the Markov renewal process $(M^{(q),+}_n, N^{(q),+}_n)_{n\geq 0}$, then we have

$$\mathbf{P}_{0,\pi^+}\left[N^{(q),+}_1\right] = \mathbf{P}_{0,\pi^+}\left[\bar{L}^{-1}_{e_q}\right] = \int_0^{+\infty} q\mathrm{e}^{-qt}\mathbf{P}_{0,\pi^+}[\bar{L}^{-1}_t]\mathrm{d}t = \frac{1}{q}c_{\pi^+}. \tag{5.18}$$

For every $\theta \in \mathcal{S}$, define

$$W^+_\theta(\mathrm{d}v, \mathrm{d}r) := \mathbf{P}_{0,\theta}\left[\int_0^{\bar{L}_\infty} \mathbb{1}_{\{\Theta^+_s\in \mathrm{d}v,\ \bar{L}^{-1}_s\in \mathrm{d}r\}}\mathrm{d}s\right]$$

and $G_\theta(\mathrm{d}v, \mathrm{d}r) := \mathbf{P}_{0,\theta}\left(M^{(q),+}_1 \in \mathrm{d}v, N^{(q),+}_1 \in \mathrm{d}r\right)$. Let $G^{*0}_\theta(\mathrm{d}v, \mathrm{d}r) := \delta_\theta(\mathrm{d}v)\delta_0(\mathrm{d}r)$ and for $n \geq 1$, let G^{*n}_θ be the nth convolution of G_θ. In view of (5.18) under the assumptions of Proposition 5.3, it follows by [2, Theorem 2.1] that

$$\lim_{t\to+\infty}\int_{\mathcal{S}\times[0,t]} g(v, t-r)\sum_{n=0}^{+\infty} G^{*n}_\theta(\mathrm{d}v, \mathrm{d}r) = \frac{q}{c_{\pi^+}}\int_{\mathcal{S}\times\mathbb{R}^+} g(v, r)\pi^+(\mathrm{d}v)\mathrm{d}r \tag{5.19}$$

for every $\theta \in \mathcal{S}$ and every measurable function $g \in \mathcal{M}$. By applying similar calculations to $W^+_\theta(\mathrm{d}v, \mathrm{d}r)$ as we did to $U^+_\theta(\mathrm{d}v, \mathrm{d}z)$, we can show that $qW^+_\theta(\mathrm{d}v, \mathrm{d}r)$ is equal to $\sum_{n=1}^{+\infty} G^{*n}_\theta(\mathrm{d}v, \mathrm{d}r)$. Hence by (5.19) we have

$$\lim_{t\to+\infty}\int_{\mathcal{S}\times[0,t]} g(v, t-r)W^+_\theta(\mathrm{d}v, \mathrm{d}r) = \frac{1}{c_{\pi^+}}\int_{\mathcal{S}\times\mathbb{R}^+} g(v, r)\pi^+(\mathrm{d}v)\mathrm{d}r. \tag{5.20}$$

LEMMA 5.2.

(i) The nondecreasing MAP $(\bar{L}^{-1}, \Theta^+)$ has a Lévy system (H^+, N^+) where $H^+_t = t \wedge \zeta^+$ and $N^+(\theta, \mathrm{d}v, \mathrm{d}r) := \Gamma^+(\theta, \mathrm{d}v, \mathrm{d}r, [0,\infty))$ is a kernel from $\mathcal{S}$ to $\mathcal{S}\times\mathbb{R}^+$.

(ii) For $r > 0$, define

$$\bar{d}_r := \inf\{s > r :\ \bar{\xi}_s = \xi_s\}.$$

Then for every $\theta \in \mathcal{S}$, $W^+_\theta(\mathrm{d}v, \mathrm{d}r)$ has a density function $w^+_\theta(\mathrm{d}v, r)$ with respect to the Lebesgue measure $\mathrm{d}r$ such that

$$\mathbf{P}_{0,\theta}\left[f(\Theta_r); \bar{d}_r = r\right] = \int_{\mathcal{S}} f(v)\ell^+(v)w^+_\theta(\mathrm{d}v, r)$$

for every nonnegative measurable function $f : \mathcal{S} \to \mathbb{R}^+$ and almost every $r > 0$. Moreover, for every $\theta \in \mathcal{S}$ and every bounded continuous function $h : \mathcal{S} \to \mathbb{R}$, the function $r \mapsto \mathbf{P}_{0,\theta}\left[h(\Theta_r); \bar{d}_r = r\right]$ is lower semi-continuous on $(0, +\infty)$.

PROOF. The claim in (i) follows by taking marginals in Proposition 2.5.

(ii) Since $t \mapsto \bar{L}_t$ is a nondecreasing and right continuous process, we have $\bar{L}_r = \inf\{s > 0 : \bar{L}^{-1}_s > r\}$ for every $r > 0$. We also note that $\bar{L}^{-1}(\bar{L}_r) = \inf\{s > r :\ \bar{\xi}_s = \xi_s\} = \bar{d}_r$. In view of this, (i) and (5.17), we can apply Proposition 2.8 to the process $(\bar{L}^{-1},\Theta^{+})$ and deduce that $1_{\{\ell^{+}(v)>0\}}W^{+}_{\theta}(\mathrm{d}v,\mathrm{d}r)$ has a density function $w^{+}_{\theta}(\mathrm{d}v,r)$ with respect to the Lebesgue measure $\mathrm{d}r$ such that

$$\mathbf{P}_{0,\theta}\left[f(\Theta_r);\bar{d}_r=r\right]=\mathbf{P}_{0,\theta}\left[f(\Theta^{+}_{\bar{L}_r});\bar{L}^{-1}(\bar{L}_r)=r\right]=\int_{\mathcal{S}}f(v)\ell^{+}(v)w^{+}_{\theta}(\mathrm{d}v,r) \tag{5.21}$$

for almost every $r>0$ and every nonnegative measurable function $f:\mathcal{S}\to\mathbb{R}^{+}$. Now take an arbitrary bounded continuous function $h:\mathcal{S}\to\mathbb{R}$. We have

$$\mathbf{P}_{0,\theta}\left[h(\Theta_r);\bar{d}_r=r\right]=\mathbf{P}_{0,\theta}\left[h(\Theta_r)\right]-\mathbf{P}_{0,\theta}\left[h(\Theta_r);\bar{d}_r>r\right].$$

It is easy to see that $r\mapsto\mathbf{P}_{0,\theta}\left[h(\Theta_r)\right]$ is right continuous on $[0,+\infty)$ since Θ is a right continuous process. We only need to show that $r\mapsto\mathbf{P}_{0,\theta}\left[h(\Theta_r);\bar{d}_r>r\right]$ is upper semi-continuous on $(0,+\infty)$. Take an arbitrary sequence $r_n\downarrow r\in(0,+\infty)$. Note that, for any $s>0$, $\bar{d}_s>s$ if and only if $s\in\cup_{g_i\in\bar{G}}[g_i,d_i)$. Hence $\{\bar{d}_{r_n}>r_n \text{ i.o.}\}=\{r_n\in\cup_{g_i\in\bar{G}}[g_i,d_i)\text{ i.o.}\}\subset\{r\in\cup_{g_i\in\bar{G}}[g_i,d_i)\}=\{\bar{d}_r>r\}$. It follows that $\limsup_{n\to+\infty}1_{\{\bar{d}_{r_n}>r_n\}}=1_{\{\bar{d}_{r_n}>r_n\text{ i.o.}\}}\le 1_{\{\bar{d}_r>r\}}$. Thus by the reverse Fatou's lemma, $\mathbf{P}_{0,\theta}\left[h(\Theta_r);\bar{d}_r>r\right]\ge\limsup_{n\to+\infty}\mathbf{P}_{0,\theta}\left[h(\Theta_{r_n});\bar{d}_{r_n}>r_n\right]$. We complete the proof. □

LEMMA 5.3. *Suppose that $((\xi,\Theta),\mathbf{P})$ and $((\xi,\Theta),\hat{\mathbf{P}})$ are a pair of upwards regular MAPs for which condition (WR) is satisfied. Under the assumptions of Proposition 5.3, we have*

(i) $\int_{\mathcal{S}}\ell^{+}(\theta)\pi^{+}(\mathrm{d}\theta)=0$, and $\int_0^{+\infty}\ell^{+}(\Theta_s)\mathrm{d}\bar{L}_s=0$, $\mathbf{P}_{0,\pi}$-a.s.
(ii) For every $y<0$,

$$\hat{H}^{+}_{\theta}(y)\pi(\mathrm{d}\theta)=\frac{1}{c_{\pi^{+}}}\int_{\mathcal{S}}\pi^{+}(\mathrm{d}\phi)\mathrm{n}^{+}_{\phi}\left(\int_0^{\zeta}1_{\{\epsilon_r\le -y,\ \nu_r\in\mathrm{d}\theta\}}\mathrm{d}r\right), \tag{5.22}$$

where $\hat{H}^{+}_{\theta}(y)=\hat{\mathbf{P}}_{y,\theta}\left(\tau^{+}_{0}=+\infty\right)$, and

$$\frac{\hat{\mathrm{n}}^{+}_{\theta}(\zeta=+\infty)}{\ell^{+}(\theta)+\mathrm{n}^{+}_{\theta}(\zeta)}\hat{U}^{+}_{\pi}(\mathrm{d}\theta,\mathbb{R}^{+})=\frac{1}{c_{\pi^{+}}}\pi^{+}(\mathrm{d}\theta). \tag{5.23}$$

PROOF. (i) By (5.3), we have

$$\int_{\mathcal{S}}\ell^{+}(\theta)\pi^{+}(\mathrm{d}\theta)=\frac{1}{\mathbf{P}_{0,\pi}\left[\bar{L}_1\right]}\mathbf{P}_{0,\pi}\left[\int_0^1\ell^{+}(\Theta_s)\mathrm{d}\bar{L}_s\right]. \tag{5.24}$$

We note that by (2.3) and Fubini's theorem,

$$\mathbf{P}_{0,\pi}\left[\int_0^{+\infty}\ell^{+}(\Theta_s)\mathrm{d}\bar{L}_s\right]=\mathbf{P}_{0,\pi}\left[\int_0^{+\infty}1_{\{s\in\bar{M}\}}\mathrm{d}s\right]=\int_0^{+\infty}\mathbf{P}_{0,\pi}\left(s\in\bar{M}\right)\mathrm{d}s.$$

By Proposition 3.3, we have for any $s>0$,

$$\mathbf{P}_{0,\pi}\left(s\in\bar{M}\right)=\mathbf{P}_{0,\pi}\left(\bar{\xi}_s-\xi_s=0\right)=\hat{\mathbf{P}}_{0,\pi}\left(\bar{\xi}_s=0\right)\le\hat{\mathbf{P}}_{0,\pi}\left(\tau^{+}_{0}\ge s\right)=0.$$

The last equality is because $((\xi,\Theta),\hat{\mathbf{P}})$ is upwards regular. It follows that

$$\mathbf{P}_{0,\pi}\left[\int_0^{+\infty}\ell^+(\Theta_s)\mathrm{d}\bar{L}_s\right]=0, \tag{5.25}$$

and hence by (5.24) $\int_{\mathcal{S}}\ell^+(\theta)\pi^+(\mathrm{d}\theta)=0$.

(ii) First we claim that

$$\mathbf{P}_{0,\pi}\left(\bar{d}_r=r\right)=0\quad\forall r>0. \tag{5.26}$$

In fact, by Lemma 5.2(ii) and (5.25), we have

$$\begin{aligned}\int_0^{+\infty}\mathbf{P}_{0,\pi}\left(\bar{d}_r=r\right)\mathrm{d}r &= \int_0^{+\infty}\mathrm{d}r\int_{\mathcal{S}}\ell^+(v)w^+_\pi(\mathrm{d}v,r)\\ &= \int_0^{+\infty}\int_{\mathcal{S}}\ell^+(v)W^+_\pi(\mathrm{d}v,\mathrm{d}r)\\ &= \mathbf{P}_{0,\pi}\left[\int_0^{+\infty}\ell^+(\Theta_s)\mathrm{d}\bar{L}_s\right]=0.\end{aligned} \tag{5.27}$$

Thus $\mathbf{P}_{0,\pi}\left(\bar{d}_r=r\right)=0$ for almost every $r>0$, and hence for every $r>0$ since $r\mapsto\mathbf{P}_{0,\pi}\left(\bar{d}_r=r\right)$ is lower semi-continuous on $(0,+\infty)$. By Proposition 3.3 we have

$$\hat{\mathbf{P}}_{0,\pi}\left[g(\Theta_0);\bar{\xi}_t\le -y\right]=\mathbf{P}_{0,\pi}\left[g(\Theta_t);\bar{\xi}_t-\xi_t\le -y\right] \tag{5.28}$$

for every $y<0$, $t\ge 0$ and every bounded measurable function $g:\mathcal{S}\to\mathbb{R}$. It follows by the bouned convergence theorem that

$$\begin{aligned}\hat{\mathbf{P}}_{0,\pi}\left[g(\Theta_0);\bar{\xi}_t\le -y\right] &= \hat{\mathbf{P}}_{0,\pi}\left[g(\Theta_0);\tau^+_{-y}>t\right]=\int_{\mathcal{S}}\pi(\mathrm{d}\theta)g(\theta)\hat{\mathbf{P}}_{0,\theta}\left(\tau^+_{-y}>t\right)\\ &\to \int_{\mathcal{S}}\pi(\mathrm{d}\theta)g(\theta)\hat{\mathbf{P}}_{0,\theta}\left(\tau^+_{-y}=+\infty\right)=\int_{\mathcal{S}}\pi(\mathrm{d}\theta)g(\theta)\hat{H}^+_\theta(y),\end{aligned} \tag{5.29}$$

as $t\to+\infty$. On the other hand, we have by (5.26)

$$\mathbf{P}_{0,\pi}\left[g(\Theta_t);\bar{\xi}_t-\xi_t\le -y\right]=\mathbf{P}_{0,\pi}\left[g(\Theta_t);\bar{\xi}_t-\xi_t\le -y,\bar{d}_t>t\right]\quad\forall t>0. \tag{5.30}$$

We note that $\bar{d}_t>t$ if and only if $t\in\cup_{g_i\in\bar{G}}[g_i,d_i)$. Hence by (2.4) the above expectation equals

$$\begin{aligned}&\mathbf{P}_{0,\pi}\left[g(\Theta_t);\bar{\xi}_t-\xi_t\le -y,t\in\cup_{g_i\in\bar{G}}[g_i,d_i)\right]\\ &\quad=\mathbf{P}_{0,\pi}\left[\int_0^t \mathrm{n}^+_{\Theta_s}\left(g(\nu_{t-s})\mathbb{1}_{\{\epsilon_{t-s}\le -y,t-s<\zeta\}}\right)\mathrm{d}\bar{L}_s\right]\\ &\quad=\mathbf{P}_{0,\pi}\left[\int_0^{+\infty}\mathbb{1}_{\{\bar{L}^{-1}_u\le t\}}\mathrm{n}^+_{\Theta^+_u}\left(g(\nu_{t-\bar{L}^{-1}_u})\mathbb{1}_{\{\epsilon_{t-\bar{L}^{-1}_u}\le -y,t-\bar{L}^{-1}_u<\zeta\}}\right)\mathrm{d}u\right]\\ &\quad=\int_{\mathcal{S}\times[0,t]}W^+_\pi(\mathrm{d}v,\mathrm{d}r)\mathrm{n}^+_v\left(g(\nu_{t-r})\mathbb{1}_{\{\epsilon_{t-r}\le -y,t-r<\zeta\}}\right).\end{aligned} \tag{5.31}$$

By (5.20), the integral in the right converges as $t \to +\infty$ to

$$\frac{1}{c_{\pi^+}} \int_{\mathcal{S}\times\mathbb{R}^+} \pi^+(\mathrm{d}v)\mathrm{d}r\, \mathrm{n}_v^+\left(g(\nu_r)\mathbb{1}_{\{\epsilon_r \le -y, r<\zeta\}}\right) = \frac{1}{c_{\pi^+}} \int_{\mathcal{S}} \pi^+(\mathrm{d}v)\mathrm{n}_v^+ \left(\int_0^\zeta g(\nu_r)\mathbb{1}_{\{\epsilon_r \le -y\}}\mathrm{d}r \right).$$

Combining this and (5.28)-(5.31) we get that

$$\int_{\mathcal{S}} \pi(\mathrm{d}\theta) g(\theta) \hat{H}_\theta^+(y) = \frac{1}{c_{\pi^+}} \int_{\mathcal{S}} \pi^+(\mathrm{d}\theta)\mathrm{n}_\theta^+ \left(\int_0^\zeta g(\nu_r)\mathbb{1}_{\{\epsilon_r \le -y\}}\mathrm{d}r \right)$$

for any bounded measurable function $g: \mathcal{S} \to \mathbb{R}$, which in turn yields (5.22).

Next we prove (5.23). It follows by Proposition 3.3 that

$$\hat{\mathbf{P}}_{0,\pi}\left[g(\bar{\Theta}_t)\right] = \mathbf{P}_{0,\pi}\left[g(\bar{\Theta}_t)\right] \quad \forall t \ge 0 \tag{5.32}$$

for any bounded measurable function $g: \mathcal{S} \to \mathbb{R}$. Similarly by (5.26) and (2.4) we have

$$\begin{aligned}
\mathbf{P}_{0,\pi}\left[g(\bar{\Theta}_t)\right] &= \mathbf{P}_{0,\pi}\left[g(\bar{\Theta}_t); t \in \cup_{g_i \in \bar{G}}[g_i, d_i)\right] \\
&= \mathbf{P}_{0,\pi}\left[\sum_{g_i \in \bar{G}} g(\Theta_{g_i})\mathbb{1}_{\{g_i \le t < d_i\}}\right] \\
&= \int_{\mathcal{S}\times[0,t]} W_\pi^+(\mathrm{d}v, \mathrm{d}r) g(v)\mathrm{n}_v^+(t-r<\zeta).
\end{aligned}$$

By (5.20), we get

$$\lim_{t\to+\infty} \mathbf{P}_{0,\pi}\left[g(\bar{\Theta}_t)\right] = \frac{1}{c_{\pi^+}} \int_{\mathcal{S}} g(\theta)\mathrm{n}_\theta^+(\zeta)\pi^+(\mathrm{d}\theta).$$

It follows by this, (5.32), the bounded convergence theorem and Lemma 5.3(i) that

$$\begin{aligned}
\hat{\mathbf{P}}_{0,\pi}\left[g(\bar{\Theta}_{e_q})\right] &= \mathbf{P}_{0,\pi}\left[g(\bar{\Theta}_{e_q})\right] \\
&= \int_0^{+\infty} \mathrm{e}^{-s}\mathbf{P}_{0,\pi}\left[g(\bar{\Theta}_{s/q})\right]\mathrm{d}s \\
&\to \frac{1}{c_{\pi^+}} \int_{\mathcal{S}} g(\theta)\mathrm{n}_\theta^+(\zeta)\pi^+(\mathrm{d}\theta) = \frac{1}{c_{\pi^+}} \int_{\mathcal{S}} g(\theta)\left(\ell^+(\theta) + \mathrm{n}_\theta^+(\zeta)\right)\pi^+(\mathrm{d}\theta)
\end{aligned} \tag{5.33}$$

as $q \to 0+$. Let $\mathcal{C}$ denote the set of nonnegative bounded measurable functions $h: \mathcal{S} \to \mathbb{R}^+$ such that $\theta \mapsto h(\theta)a^+(\theta)/\left(\ell^+(\theta) + \mathrm{n}_\theta^+(\zeta)\right)$ is a bounded function. On the one hand, by (5.33) we have

$$\hat{\mathbf{P}}_{0,\pi}\left[\frac{h(\bar{\Theta}_{e_q})a^+(\bar{\Theta}_{e_q})}{\ell^+(\bar{\Theta}_{e_q}) + \mathrm{n}^+_{\bar{\Theta}_{e_q}}(\zeta)}\right] \to \frac{1}{c_{\pi^+}} \int_{\mathcal{S}} h(\theta)a^+(\theta)\pi^+(\mathrm{d}\theta) \quad \text{as } q \to 0+ \tag{5.34}$$

for any $h \in \mathcal{C}$. On the other hand, by Proposition 2.3 we have

(5.35)

$$\hat{\mathbf{P}}_{0,\pi}\left[\frac{h(\bar{\Theta}_{e_q})a^+(\bar{\Theta}_{e_q})}{\ell^+(\bar{\Theta}_{e_q}) + \mathrm{n}^+_{\bar{\Theta}_{e_q}}(\zeta)}\right] = \int_{\mathcal{S}\times\mathbb{R}^+} \hat{W}_\pi^+(\mathrm{d}v, \mathrm{d}r)\mathrm{e}^{-qr}\frac{h(v)a^+(v)}{\ell^+(v) + \mathrm{n}_v^+(\zeta)}\left(q\hat{\ell}^+(v) + \hat{\mathrm{n}}_v^+(1-\mathrm{e}^{-q\zeta})\right).$$

If we can show that
(5.36)

$$\lim_{q\to 0+}\hat{\mathbf{P}}_{0,\pi}\left[\frac{h(\bar\Theta_{e_q})a^+(\bar\Theta_{e_q})}{\ell^+(\bar\Theta_{e_q})+\mathrm{n}^+_{\bar\Theta_{e_q}}(\zeta)}\right]=\int_{\mathcal{S}}\hat W^+_\pi(\mathrm{d}v,\mathbb{R}^+)\frac{h(v)a^+(v)}{\ell^+(v)+\mathrm{n}^+_v(\zeta)}\hat{\mathrm{n}}^+_v(\zeta=+\infty),\quad \forall h\in\mathcal{C},$$

then by (5.34) and the fact that $\hat W^+_\pi(\mathrm{d}v,\mathbb{R}^+)=\hat U^+_\pi(\mathrm{d}v,\mathbb{R}^+)=\hat{\mathbf{P}}_{0,\pi}\left[\int_0^{\bar L_\infty}\mathbb{1}_{\{\hat\Theta^+_s\in\mathrm{d}v\}}\mathrm{d}s\right]$ we get

$$\int_{\mathcal{S}}\hat U^+_\pi(\mathrm{d}v,\mathbb{R}^+)\frac{h(v)a^+(v)}{\ell^+(v)+\mathrm{n}^+_v(\zeta)}\hat{\mathrm{n}}^+_v(\zeta=+\infty)=\frac{1}{c_{\pi^+}}\int_{\mathcal{S}}h(\theta)a^+(\theta)\pi^+(\mathrm{d}\theta)\quad \forall h\in\mathcal{C}. \tag{5.37}$$

Note that for any $q\in(0,1]$ the integrand in the right of (5.35) is bounded from above by

$$\|h\|_\infty\frac{a^+(v)}{\ell^+(v)+\mathrm{n}^+_v(\zeta)}\left(\hat\ell^+(v)+\hat{\mathrm{n}}^+_v(1-\mathrm{e}^{-\zeta})\right).$$

Hence to prove (5.36) it suffices to prove

$$\int_{\mathcal{S}}\hat W^+_\pi(\mathrm{d}v,\mathbb{R}^+)\frac{a^+(v)}{\ell^+(v)+\mathrm{n}^+_v(\zeta)}\left(\hat\ell^+(v)+\hat{\mathrm{n}}^+_v(1-\mathrm{e}^{-\zeta})\right)<+\infty. \tag{5.38}$$

By Proposition 3.3 and Proposition 2.3 the above integral is equal to

$$\begin{aligned}\hat{\mathbf{P}}_{0,\pi}\left[\mathrm{e}^{\bar g_{e_1}}\frac{a^+(\bar\Theta_{e_1})}{\ell^+(\bar\Theta_{e_1})+\mathrm{n}^+_{\bar\Theta_{e_1}}(\zeta)}\right] &= \mathbf{P}_{0,\pi}\left[\mathrm{e}^{(e_1-\bar g_{e_1})}\frac{a^+(\bar\Theta_{e_1})}{\ell^+(\bar\Theta_{e_1})+\mathrm{n}^+_{\bar\Theta_{e_1}}(\zeta)}\right]\\ &= \int_{\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^+}\mathrm{e}^{-r}a^+(v)V^+_\pi(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z)\\ &= \mathbf{P}_{0,\pi}\left[\int_0^{+\infty}\mathrm{e}^{-\bar L^{-1}_s}a^+(\Theta^+_s)\mathrm{d}s\right]\\ &= \mathbf{P}_{0,\pi}\left[\int_0^{\bar L_{e_1}}a^+(\Theta^+_s)\mathrm{d}s\right].\end{aligned}$$

The finiteness of the final expectation is implied by the finiteness of $\mathbf{P}_{0,\pi^+}\left[\xi^+_1\right]$. Indeed, by (5.3) and Markov property

$$\begin{aligned}\mathbf{P}_{0,\pi^+}\left[\xi^+_1\right] &= \frac{1}{\mathbf{P}_{0,\pi}\left[\bar L_1\right]}\mathbf{P}_{0,\pi}\left[\int_0^1\mathbf{P}_{0,\Theta_s}\left[\xi^+_1\right]\mathrm{d}\bar L_s\right]\\ &= \frac{1}{\mathbf{P}_{0,\pi}\left[\bar L_1\right]}\mathbf{P}_{0,\pi}\left[\int_0^{\bar L_1}\mathbf{P}_{0,\Theta^+_s}\left[\xi^+_1\right]\mathrm{d}s\right]\\ &= \frac{1}{\mathbf{P}_{0,\pi}\left[\bar L_1\right]}\mathbf{P}_{0,\pi}\left[\int_0^{\bar L_1}\left(\xi^+_{s+1}-\xi^+_s\right)\mathrm{d}s\right].\end{aligned}$$

Since the continuous part of $\xi^+_{s+1}-\xi^+_s$ is $\int_s^{s+1}a^+(\Theta^+_r)\mathrm{d}r$, we get by Fubini's theorem that

$$+\infty>\mathbf{P}_{0,\pi^+}\left[\xi^+_1\right]\mathbf{P}_{0,\pi}\left[\bar L_1\right]\geq\mathbf{P}_{0,\pi}\left[\int_0^{\bar L_1}\mathrm{d}s\int_s^{s+1}a^+(\Theta^+_r)\mathrm{d}r\right]=\mathbf{P}_{0,\pi}\left[\int_0^{\bar L_1+1}a^+(\Theta^+_r)\mathrm{d}r\right].$$

By writing $\mathbf{P}_{0,\pi}\left[\int_0^{\bar{L}_s} a^+(\Theta_r^+)\mathrm{d}r\right] = \mathbf{P}_{0,\pi}\left[\int_0^s a^+(\Theta_r)\mathrm{d}\bar{L}_r\right]$, one can easily show that $s \mapsto \mathbf{P}_{0,\pi}\left[\int_0^{\bar{L}_s} a^+(\Theta_r^+)\mathrm{d}r\right]$ is subadditive and locally bounded nonnegative function, which in turn implies that $\mathbf{P}_{0,\pi}\left[\int_0^{\bar{L}_{e_1}} a^+(\Theta_s^+)\mathrm{d}s\right] < +\infty$.

We deduce therefrom that (5.36) and hence (5.37) hold for every $h \in \mathcal{C}$. Now, for a general nonnegative measurable function h, one can always find an nondecreasing sequence of functions $h_n \in \mathcal{C}$ such that $h_n \to h$ in the pointwise sense. Using this and the monotone convergence theorem, one gets that (5.37) holds for any nonnegative function h. The identity (5.23) follows immediately. □

PROPOSITION 5.4. *Suppose that the assumptions of Lemma 5.3 hold. Then the stationary distribution $\rho^{\oplus}(\mathrm{d}y, \mathrm{d}v)$ given in Proposition 5.3 can be represented by*

$$\rho^{\oplus}(\mathrm{d}y, \mathrm{d}v) = \rho_1^{\oplus}(\mathrm{d}y, \mathrm{d}v) + \rho_2^{\oplus}(\mathrm{d}y, \mathrm{d}v)$$

where

$$\rho_1^{\oplus}(\mathrm{d}y, \mathrm{d}v) := \frac{c_{\pi^+}}{\mu^+} \mathbb{1}_{\{y<0\}} \bar{\Pi}_v(-y) \hat{H}_v^+(y) \mathrm{d}y \, \pi(\mathrm{d}v),$$

and

$$\rho_2^{\oplus}(\mathrm{d}y, \mathrm{d}v) := \frac{c_{\pi^+}}{\mu^+} \frac{a^+(v)\hat{\mathrm{n}}_v^+(\zeta = +\infty)}{\ell^+(v) + \mathrm{n}_v^+(\zeta)} \delta_0(\mathrm{d}y) \hat{U}_{\pi}^+(\mathrm{d}v, \mathbb{R}^+).$$

Part III

Main results and their proofs

6. Assumptions and main results. Recall that $(X, \{\mathbb{P}_z, z \in \mathcal{H}\})$ is an $\mathcal{H}$-valued ssMp and $((\xi, \Theta), \mathbf{P})$ is the corresponding MAP via the Lamperti-Kiu transform, for which we have assumed its Lévy system (H, Π) satisfies $H_t = t$ until killing. We assume the following additional conditions hold.

- (a1) $(X, \{\mathbb{P}_z, z \in \mathcal{H}\})$ is a Feller process.
- (a2) The modulator of $((\xi, \Theta), \mathbf{P})$ is a positive recurrent process having an invariant distribution π which is fully supported on $\mathcal{S}$. The continuous part of ξ^+ of $((\xi^+, \Theta^+), \mathbf{P})$ can be represented by $\int_0^t a^+(\Theta_s^+)\mathrm{d}s$ for some strictly positive measurable function a^+ on $\mathcal{S}$.
- (a3) $((\xi, \Theta), \mathbf{P})$ and $((\xi, \Theta), \hat{\mathbf{P}})$ are a pair of upwards regular MAPs for which (WR) is satisfied.
- (a4) $((\xi, \Theta), \hat{\mathbf{P}})$ satisfies condition (HT).
- (a5) $\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,1]} |\xi_1|\right] < +\infty$.
- (a6) The modulator of the ascending ladder height process $((\xi^+, \Theta^+), \mathbf{P})$ is a nonarithmetic aperiodic Harris recurrent process having an invariant distribution π^+ on $\mathcal{S}$ with full support such that $\mathbf{P}_{0,\pi^+}\left[\xi_1^+\right] < +\infty$.

(a7) $\hat{\mathrm{n}}_v^+(\zeta=+\infty)>0$ for every $v\in\mathcal{S}$.
(a8) $\inf_{v\in\mathcal{S}}\left[\ell^+(v)+\mathrm{n}_v^+\left(1-\mathrm{e}^{-\zeta}\right)\right]>0$ and $\mathrm{n}_v^+(\zeta)<+\infty$ for every $v\in\mathcal{S}$.

As noted in Section 5, given conditions (a2) and (a8), it follows by Corollary 2.15 that

$$\pi^+(\cdot)=\frac{1}{\mathbf{P}_{0,\pi}\left[\bar{L}_1\right]}\mathbf{P}_{0,\pi}\left[\int_0^1 \mathbb{1}_{\{\Theta_s\in\cdot\}}\mathrm{d}\bar{L}_s\right]$$

is an invariant distribution for Θ^+. Moreover, the Harris recurrence of Θ^+ given in (a6) implies that π^+ is the unique invariant distribution for Θ^+.

THEOREM 6.1. *Under assumptions (a1)-(a8), the conclusions (C1)-(C5) in the Introduction are true.*

We conclude this section by considering a slight adjustment of the sufficient conditions (a1)-(a8), such that (a5) and (a7) can be replaced by the stronger sufficient conditions (i.e. ones that imply (a5) and (a7)). Our principal aim here is to produce conditions that can be identified in terms of the components of the ascending ladder process of $((\xi,\Theta),\mathbf{P})$ and the ascending ladder process of the dual process $((\xi,\Theta),\hat{\mathbf{P}})$. More precisely, we have the following alternative conditions to Theorem 6.1.

THEOREM 6.2. *Suppose conditions (a5) and (a7) are replaced by:*

(a5)' The modulator $(\Theta_t^+)_{t\geq 0}$ of the ascending ladder height process $((\xi^+,\Theta^+),\hat{\mathbf{P}})$, is an aperiodic Harris recurrent process having an invariant distribution $\hat{\pi}^+$ on $\mathcal{S}$ with full support such that $\int_{\mathcal{S}}\hat{\pi}^+(\mathrm{d}v)\left[\hat{a}^+(v)+\hat{\mathrm{n}}_v^+(|\epsilon_\zeta|;\zeta<\infty)\right]<+\infty$.

(a7)' $\inf_{v\in\mathcal{S}}\hat{\mathrm{n}}_v^+(\zeta=+\infty)>0$.

Then the conclusion of Theorem 6.1 is still valid.

REMARK 6.3. Before continuing to the proof, let us note that the condition in (a5)' is the natural analogue of (a6). Indeed, note that $\mathbf{P}_{0,\pi^+}\left[\xi_1^+\right]=\int_{\mathcal{S}}\pi^+(\mathrm{d}v)\left[a^+(v)+\mathrm{n}_v^+(|\epsilon_\zeta|;\zeta<\infty)\right]$.

PROOF OF THEOREM 6.2. Condition (a7)' obviously implies (a7). The proof is based around showing that the new conditions together with (a1)-(a4) and (a8) imply (a5). Suppose that $\mathbf{e}_q$ is an independent exponentially distributed random variable with rate $q>0$. On account of the fact that $t\mapsto\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,t]}|\xi_s|\right]$ is increasing, to show (a5) it suffices to show that

$$\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,\mathbf{e}_q]}|\xi_s|\right]=\int_0^\infty q\mathrm{e}^{-qt}\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,t]}|\xi_s|\right]\mathrm{d}t<\infty.$$

For the latter, we note that

$$\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,\mathbf{e}_q]}|\xi_s|\right]\leq\mathbf{P}_{0,\pi}\left[\bar{\xi}_{\mathbf{e}_q}\right]-\mathbf{P}_{0,\pi}\left[\underline{\xi}_{\mathbf{e}_q}\right].$$

Next define

$$\Lambda_v^+(q) := \ell^+(v)q + \mathrm{n}_v^+(1-\mathrm{e}^{-q\zeta}), \qquad q \geq 0. \tag{6.1}$$

Note from Proposition 2.3 that

$$\mathbf{P}_{0,\pi}\left[\bar{\xi}_{\mathbf{e}_q}\right] = \mathbf{P}_{0,\pi}\left[\int_0^\infty \mathbb{1}_{\{\xi_s^+<\infty\}}\mathrm{e}^{-q\bar{L}_s^{-1}}\xi_s^+\Lambda_{\Theta_s^+}^+(q)\mathrm{d}s\right]. \tag{6.2}$$

Next define the change of measure

$$\left.\frac{\mathrm{d}\mathbf{P}_{0,\theta}^{(q)}}{\mathrm{d}\mathbf{P}_{0,\theta}}\right|_{\mathcal{G}_t} = \mathrm{e}^{-q\bar{L}_t^{-1}+\int_0^t \Lambda_{\Theta_s^+}^+(q)\mathrm{d}s} \tag{6.3}$$

for $\theta \in \mathcal{S}$, where $\mathcal{G}_t = \sigma((\bar{L}_s^{-1}, \xi_s^+, \Theta_s^+), s \leq t)$. To see why the right-hand side of (6.3) is a martingale, it suffices to note that $(\bar{L}_t^{-1}, \Theta_t^+)_{t\geq 0}$ is a MAP and that, for $\theta \in \mathcal{S}$,

$$\mathbf{P}_{0,\theta}[\mathrm{e}^{-q\bar{L}_t^{-1}}|\Theta_s^+ : s \leq t] = \mathrm{e}^{-\int_0^t \Lambda_{\Theta_s^+}^+(q)\mathrm{d}s}, \qquad t \geq 0,$$

which follows from the the definition (6.1) and the fact that the constituent parts of Λ_v^+, namely $\ell^+(v)$ and $\mathrm{n}_v^+(1-\mathrm{e}^{-q\zeta})$ describe the rate at which $\bar{L}_s^{-1}$ moves continuously and with jumps given $\Theta_s^+ = v$, for $v \in \mathcal{S}$.

Using (6.3) in (6.2), we have

$$\mathbf{P}_{0,\pi}\left[\bar{\xi}_{\mathbf{e}_q}\right] = \mathbf{P}_{0,\pi}^{(q)}\left[\int_0^\infty \mathrm{e}^{-\int_0^s \Lambda_{\Theta_u^+}^+(q)\mathrm{d}u}\xi_s^+\Lambda_{\Theta_s^+}(q)\mathrm{d}s\right] = \mathbf{P}_{0,\pi}^{(q)}\left[\int_0^\infty \mathrm{e}^{-\int_0^s \Lambda_{\Theta_u^+}^+(q)\mathrm{d}u}\mathrm{d}\xi_s^+\right],$$

where the final equality follows by a straightforward integration by parts (recall that the process ξ^+ is non-decreasing and therefore has bounded variation paths). From (a8), we now have that there exists a constant $c > 0$ such that for any $q \geq 1$

$$\mathbf{P}_{0,\pi}\left[\bar{\xi}_{\mathbf{e}_q}\right] \leq \mathbf{P}_{0,\pi}^{(q)}\left[\int_0^\infty \mathrm{e}^{-cs}\mathrm{d}\xi_s^+\right] = c\mathbf{P}_{0,\pi}^{(q)}\left[\int_0^\infty \mathrm{e}^{-cs}\xi_s^+\mathrm{d}s\right] = c\int_0^\infty \mathrm{e}^{-cs}\mathbf{P}_{0,\pi}^{(q)}\left[\xi_s^+\right]\mathrm{d}s, \tag{6.4}$$

where, again, we have performed an integration by parts. Next note that, given Θ^+, the exponent associated to $(\bar{L}_t^{-1}, \xi_t^+)_{t\geq 0}$, is given by

$$\mathbf{P}_{0,\pi}^{(q)}[\mathrm{e}^{-\vartheta\bar{L}_t^{-1}-\beta\xi_t^+}|\Theta^+] = \exp\left\{-\int_0^t \mathrm{d}s\left[\vartheta\ell^+(\Theta_s^+) + \beta a^+(\Theta_s^+) + \mathrm{n}_{\Theta_s^+}^+((1-\mathrm{e}^{-\vartheta\zeta-\beta\epsilon_\zeta})\mathrm{e}^{-q\zeta}; \zeta < \infty)\right]\right\},$$

for $\vartheta, \beta, t \geq 0$. From this it is easily deduced by differentiation that

$$\begin{aligned}\mathbf{P}^{(q)}[\xi_t^+|\Theta^+] &= \int_0^t \mathrm{d}s\left[a^+(\Theta_s^+) + \mathrm{n}_{\Theta_s^+}^+(|\epsilon_\zeta|\mathrm{e}^{-q\zeta}; \zeta < \infty)\right] \\ &\leq \int_0^t a^+(\Theta_s^+) + \mathrm{n}_{\Theta_s^+}^+(|\epsilon_\zeta|; \zeta < \infty)\mathrm{d}s \\ &= \mathbf{P}[\xi_t^+|\Theta^+].\end{aligned}$$

Using the ergodic properties of Θ^+ under $\mathbf{P}$, we can invoke Theorem 1.1. of [19] and conclude that

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\mathbf{P}^{(q)}_{0,\pi}[\xi^+_t] &\le \lim_{t\to\infty}\frac{1}{t}\mathbf{P}_{0,\pi}[\xi^+_t]\\ &= \lim_{t\to\infty}\frac{1}{t}\mathbf{P}_{0,\pi}\left[\int_0^t a^+(\Theta^+_s)+\mathrm{n}^+_{\Theta^+_s}(|\epsilon_\zeta|;\zeta<\infty)\mathrm{d}s\right]\\ &= \int_{\mathcal{S}}\pi^+(\mathrm{d}v)\left[a^+(v)+\mathrm{n}^+_v(|\epsilon_\zeta|;\zeta<\infty)\right]\\ &= \mathbf{P}_{0,\pi^+}[\xi^+_1]\end{aligned}$$

Using the above linear growth, it follows from (6.4) that $\mathbf{P}_{0,\pi}\left[\bar{\xi}_{\mathbf{e}_q}\right]<\infty$.

Using obvious notation, the analogous object to $\Lambda^+_v(q)$ for the descending ladder height MAP takes the form

$$\Lambda^-_v(q)=\mathrm{n}^-_v(\zeta=+\infty)+\ell^-(v)q+\mathrm{n}^-_v(1-\mathrm{e}^{-q\zeta};\zeta<\infty),\qquad q\ge 0$$

(Specifically, we cannot rule out the possibility of killing.) Let us momentarily assume that the modulator of the descending ladder height process $((\xi^-,\Theta^-),\mathbf{P})$ is an aperiodic Harris recurrent process with an invariant distribution π^- on $\mathcal{S}$ with full support such that $\int_{\mathcal{S}}\pi^-(\mathrm{d}v)\left[a^-(v)+\mathrm{n}^-_v(|\epsilon_\zeta|;\zeta<\infty)\right]<+\infty$ and $\inf_{v\in\mathcal{S}}\mathrm{n}^-_v(\zeta=+\infty)>0$. Following the above computations, albeit using the last lower bound to justify the lower bounding constant c in (6.4), we can show that $\mathbf{P}_{0,\pi}\left[\underline{\xi}_{\mathbf{e}_q}\right]<\infty$.

To complete the proof, we need to show that the assumptions in the last paragraph match those in the statement of the theorem by verifying that $\mathbf{P}_{0,\pi^-}\left[\xi^-_1\right]=\hat{\mathbf{P}}_{0,\hat{\pi}^+}\left[\xi^+_1\right]$. Thanks to the weak reversal relation between $\mathbf{P}$ and $\tilde{\mathbf{P}}$ (see the discussion below Lemma 3.2), we have that $\mathbf{P}_{0,\pi^-}[\xi^-_1]=\tilde{\mathbf{P}}_{0,\tilde{\pi}^-}[\xi^-_1]$, where $\tilde{\pi}^-$ plays the role of π^- but for $((\xi,\Theta),\widetilde{\mathbf{P}})$. The relation between $\tilde{\mathbf{P}}$ and $\hat{\mathbf{P}}$ then implies that $\tilde{\pi}^-=\hat{\pi}^+$ and $\tilde{\mathbf{P}}_{0,\tilde{\pi}^-}[\xi^-_1]=\hat{\mathbf{P}}_{0,\pi^+}[\xi^+_1]$ as required. □

The remainder of the paper is devoted to the proof of Theorem 6.1. Hereafter we always assume conditions (a1)-(a8) hold unless otherwise stated.

7. Construction of entrance law. We define the killed process $(\xi^\dagger,\Theta^\dagger)$ by setting

$$(\xi^\dagger_t,\Theta^\dagger_t):=\begin{cases}(\xi_t,\Theta_t) & \text{if } t<\tau^+_0\\ \partial & \text{if } t\ge\tau^+_0.\end{cases}$$

The next lemma is the analogue of Hunt's switching identity (see [6, Theorem II.5] for the case of Lévy processes). It follows from the proof of [21, Theorem(11.3)], we include it here for completeness.

LEMMA 7.1. *$((\xi^\dagger,\Theta^\dagger),\hat{\mathbf{P}})$ and $((\xi^\dagger,\Theta^\dagger),\mathbf{P})$ are dual with respect to* Leb $\otimes\,\pi$.

Proof. Let $\mu := \mathrm{Leb}\otimes\pi$ and fix an arbitrary $t>0$. Then from Proposition 3.2 and Lemma 3.1 we see that the process $((\xi_{(t-s)-},\Theta_{(t-s)-})_{s\le t},\mathbf{P}_\mu)$ has the same law as $((\xi_s,\Theta_s)_{s\le t},\hat{\mathbf{P}}_\mu)$. It follows that the triple process $((\xi_{(t-s)-},\Theta_{(t-s)-},\bar{\xi}_t)_{s\le t},\mathbf{P}_\mu)$ has the same law as $((\xi_s,\Theta_s,\bar{\xi}_t)_{s\le t},\hat{\mathbf{P}}_\mu)$. Thus for any nonnegative measurable functions $f,g:\mathbb{R}\times\mathcal{S}\to\mathbb{R}^+$,

$$\begin{aligned}\int_{\mathbb{R}\times\mathcal{S}}\mu(\mathrm{d}y,\mathrm{d}\theta)g(y,\theta)\hat{\mathbf{P}}_{y,\theta}[f(\xi_t^\dagger,\Theta_t^\dagger)] &= \int_{\mathbb{R}\times\mathcal{S}}\mu(\mathrm{d}y,\mathrm{d}\theta)\hat{\mathbf{P}}_{y,\theta}[g(\xi_0,\Theta_0)f(\xi_t,\Theta_t)\mathbb{1}_{\{\bar{\xi}_t\le 0\}}]\\ &= \int_{\mathbb{R}\times\mathcal{S}}\mu(\mathrm{d}y,\mathrm{d}\theta)\mathbf{P}_{y,\theta}[g(\xi_{t-},\Theta_{t-})f(\xi_0,\Theta_0)\mathbb{1}_{\{\bar{\xi}_t\le 0\}}]\\ &= \int_{\mathbb{R}\times\mathcal{S}}\mu(\mathrm{d}y,\mathrm{d}\theta)\mathbf{P}_{y,\theta}[g(\xi_t,\Theta_t)f(\xi_0,\Theta_0)\mathbb{1}_{\{\bar{\xi}_t\le 0\}}]\\ &= \int_{\mathbb{R}\times\mathcal{S}}\mu(\mathrm{d}y,\mathrm{d}\theta)f(y,\theta)\mathbf{P}_{y,\theta}[g(\xi_t^\dagger,\Theta_t^\dagger)],\end{aligned}$$

where in the third equality we have used the quasi-left continuity of $((\xi,\Theta),\mathbf{P})$. □

Recall the definition of φ from (1.2). Let us define the time-changed process $(\xi^\varphi,\Theta^\varphi)$ by setting

$$(\xi_t^\varphi,\Theta_t^\varphi) := (\xi_{\varphi(t)},\Theta_{\varphi(t)}) \qquad \forall\ 0\le t<\bar{\zeta},$$

where $\bar{\zeta} := \int_0^\infty \exp\{\alpha\xi_u\}\,\mathrm{d}u$ is the lifetime of $(\xi^\varphi,\Theta^\varphi)$. We denote by $(\xi^{\varphi,\dagger},\Theta^{\varphi,\dagger})$ the process of $(\xi^\varphi,\Theta^\varphi)$ killed after the time $\tau_0^{\varphi,+} := \inf\{t\ge 0:\xi_t^\varphi>0\}$.

Lemma 7.2. *The processes* $((\xi^{\varphi,\dagger},\Theta^{\varphi,\dagger}),\mathbf{P})$ *and* $((\xi^\varphi,\Theta^\varphi),\hat{\mathbf{P}}^\downarrow)$ *are dual with respect to the measure*

$$\nu_0(\mathrm{d}y,\mathrm{d}\theta) := \mathbb{1}_{\{y<0\}}\frac{c_{\pi^+}}{\mu^+}\,\mathrm{e}^{\alpha y}\hat{H}_\theta^+(y)\mathrm{d}y\pi(\mathrm{d}\theta).$$

Proof. Let $f,g:\mathbb{R}\times\mathcal{S}\to\mathbb{R}^+$ be two nonnegative measurable functions. By the definition of $\hat{\mathbf{P}}^\downarrow$ given in Section 4 we have

$$\begin{aligned}&\int_{(-\infty,0)\times\mathcal{S}}\mathrm{d}y\pi(\mathrm{d}\theta)\,\hat{H}_\theta^+(y)g(y,\theta)\hat{\mathbf{P}}_{y,\theta}^\downarrow[f(\xi_t,\Theta_t)]\\ &= \int_{(-\infty,0)\times\mathcal{S}}\mathrm{d}y\pi(\mathrm{d}\theta)\hat{H}_\theta^+(y)g(y,\theta)\hat{\mathbf{P}}_{y,\theta}\left[f(\xi_t,\Theta_t)\frac{\hat{H}_{\Theta_t}^+(\xi_t)}{\hat{H}_\theta^+(y)};t<\tau_0^+\right]\\ &= \int_{(-\infty,0)\times\mathcal{S}}\mathrm{d}y\pi(\mathrm{d}\theta)g(y,\theta)\hat{\mathbf{P}}_{y,\theta}\left[f(\xi_t,\Theta_t)\hat{H}_{\Theta_t}^+(\xi_t);t<\tau_0^+\right]\\ &= \int_{(-\infty,0)\times\mathcal{S}}\mathrm{d}y\pi(\mathrm{d}\theta)\hat{H}_\theta^+(y)f(y,\theta)\mathbf{P}_{y,\theta}\left[g(\xi_t,\Theta_t);t<\tau_0^+\right].\end{aligned} \tag{7.1}$$

In the final equality we have applied Lemma 7.1. The above equations show that $((\xi^\dagger,\Theta^\dagger),\mathbf{P})$ and $((\xi,\Theta),\hat{\mathbf{P}}^\downarrow)$ are dual with respect to the measure

$$\mu(\mathrm{d}y,\mathrm{d}\theta) := \mathbb{1}_{\{y<0\}}\frac{c_{\pi^+}}{\mu^+}\hat{H}_\theta^+(y)\mathrm{d}y\pi(\mathrm{d}\theta).$$

Next for $t \geq 0$, define

$$A_t := \int_0^t \exp\{\alpha \xi_u\} \, \mathrm{d}u.$$

Then A_t is an additive functional in the sense that

$$A_{t+s} = A_t + A'_t \circ \theta_s \qquad t, s \geq 0$$

where θ is the shift operator and A' is an independent copy of A. Since φ is the right inverse of A, [31, Theorem 4.5] states that the time-changed processes $((\xi^{\varphi,\dagger}, \Theta^{\varphi,\dagger}), \mathbf{P})$ and $((\xi^{\varphi}, \Theta^{\varphi}), \hat{\mathbf{P}}^{\downarrow})$ are dual with respect to the Revuz measure ν associated with A_t, which is determined by the following formula:

$$\int_{\mathbb{R}\times\mathcal{S}} f(y,\theta)\nu(\mathrm{d}y, \mathrm{d}\theta) = \lim_{t\to 0+} \frac{1}{t} \int_{\mathbb{R}\times\mathcal{S}} \mu(\mathrm{d}z, \mathrm{d}v)\mathbf{P}_{z,v} \left[\int_0^t f(\xi_s^\dagger, \Theta_s^\dagger) dA_s \right] \tag{7.2}$$

for every nonnegative measurable functions $f : \mathbb{R} \times \mathcal{S} \to \mathbb{R}^+$. By Fubini's theorem and the duality relation obtained in (7.1) we have

$$\begin{aligned}
\text{RHS of (7.2)} &= \lim_{t\to 0+} \frac{1}{t} \int_{\mathbb{R}\times\mathcal{S}} \mu(\mathrm{d}z, \mathrm{d}v)\mathbf{P}_{z,v} \left[\int_0^t f(\xi_s^\dagger, \Theta_s^\dagger) \mathrm{e}^{\alpha\xi_s^\dagger} \mathrm{d}s \right] \\
&= \lim_{t\to 0+} \int_{\mathbb{R}\times\mathcal{S}} \mu(\mathrm{d}z, \mathrm{d}v) \mathrm{e}^{\alpha z} f(z,v) \frac{1}{t} \int_0^t \hat{\mathbf{P}}^{\downarrow}_{z,v}(s < \zeta) \mathrm{d}s \\
&= \int_{\mathbb{R}\times\mathcal{S}} \mu(\mathrm{d}z, \mathrm{d}v) \mathrm{e}^{\alpha z} f(z,v).
\end{aligned}$$

In the final equality we use the dominated convergence theorem. Hence the processes $((\xi^{\varphi,\dagger}, \Theta^{\varphi,\dagger}), \mathbf{P})$ and $((\xi^{\varphi}, \xi^{\varphi}), \hat{\mathbf{P}}^{\downarrow})$ are dual with respect to $\mathrm{e}^{\alpha y}\mu(\mathrm{d}y, \mathrm{d}\theta) = \mathbb{1}_{\{y<0\}} \frac{c_{\pi^+}}{\mu^+} \mathrm{e}^{\alpha y} \hat{H}^+_\theta(y) \mathrm{d}y \pi(\mathrm{d}\theta)$. $\square$

Now we wish to apply Lemma 3.2 to the dual processes $((\xi^{\varphi,\dagger}, \Theta^{\varphi,\dagger}), \mathbf{P})$ and $((\xi^{\varphi}, \Theta^{\varphi}), \hat{\mathbf{P}}^{\downarrow})$. In order to do so, we need to check the integral condition given in Lemma 3.2. We will show the integral condition in Lemma 3.2 by breaking it up into two lemmas as follows.

Lemma 7.3. *For every nonnegative measurable function $f : \mathbb{R} \times \mathcal{S} \to \mathbb{R}^+$,*

$$\int_{\mathbb{R}\times\mathcal{S}} \rho_1^{\oplus}(\mathrm{d}y, \mathrm{d}\theta) \hat{\mathbf{P}}^{\downarrow}_{y,\theta} \left[\int_0^{\bar{\zeta}} f(\xi_t^\varphi, \Theta_t^\varphi) \, \mathrm{d}t \right] = \int_{\mathbb{R}\times\mathcal{S}} \nu_0(\mathrm{d}y, \mathrm{d}\theta) f(y,\theta) \mathbf{P}_{y,\theta}(\xi_{\tau_0^+} > 0).$$

Proof. Let $f:\mathbb{R}\times\mathcal{S}\to\mathbb{R}$ be an arbitrary nonnegative measurable function. We have

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathcal{S}}\rho_1^{\oplus}(\mathrm{d}y,\mathrm{d}\theta)\hat{\mathbf{P}}^{\downarrow}_{y,\theta}\left[\int_0^{\bar\zeta} f(\xi^\varphi_t,\Theta^\varphi_t)\,\mathrm{d}t\right]\\
&=\frac{c_{\pi^+}}{\mu^+}\int_{(-\infty,0)\times\mathcal{S}}\mathrm{d}y\mathrm{e}^{\alpha y}\hat{H}^+_\theta(y)\pi(\mathrm{d}\theta)\mathrm{e}^{-\alpha y}\bar\Pi_\theta(-y)\hat{\mathbf{P}}^{\downarrow}_{y,\theta}\left[\int_0^{\bar\zeta} f(\xi^\varphi_t,\Theta^\varphi_t)\,\mathrm{d}t\right]\\
&=\int_{\mathbb{R}\times\mathcal{S}}\nu_0(\mathrm{d}y,\mathrm{d}\theta)\mathrm{e}^{-\alpha y}\bar\Pi_\theta(-y)\hat{\mathbf{P}}^{\downarrow}_{y,\theta}\left[\int_0^{\bar\zeta} f(\xi^\varphi_t,\Theta^\varphi_t)\,\mathrm{d}t\right]\\
&=\int_{\mathbb{R}\times\mathcal{S}}\nu_0(\mathrm{d}y,\mathrm{d}\theta)f(y,\theta)\mathbf{P}_{y,\theta}\left[\int_0^{\tau_0^{\varphi,+}}\mathrm{e}^{-\alpha\xi^\varphi_t}\bar\Pi_{\Theta^\varphi_t}(-\xi^\varphi_t)\mathrm{d}t\right],
\end{aligned}$$

where $\bar\Pi_v(z)=\Pi(v,\mathcal{S},(z,+\infty))$. The last equality follows from Lemma 7.2. We undo the time-change and write

$$\mathbf{P}_{y,\theta}\left[\int_0^{\tau_0^{\varphi,+}}\mathrm{e}^{-\alpha\xi^\varphi_t}\bar\Pi_{\Theta^\varphi_t}(-\xi^\varphi_t)\,\mathrm{d}t\right]=\mathbf{P}_{y,\theta}\left[\int_0^{\tau_0^+}\bar\Pi_{\Theta_t}(-\xi_t)\,\mathrm{d}t\right].$$

Hence we get

(7.3)
$$\int_{\mathbb{R}\times\mathcal{S}}\rho_1^{\oplus}(\mathrm{d}y,\mathrm{d}\theta)\hat{\mathbf{P}}^{\downarrow}_{y,\theta}\left[\int_0^{\bar\zeta} f(\xi^\varphi_t,\Theta^\varphi_t)\,\mathrm{d}t\right]=\int_{\mathbb{R}\times\mathcal{S}}\nu_0(\mathrm{d}y,\mathrm{d}\theta)f(y,\theta)\mathbf{P}_{y,\theta}\left[\int_0^{\tau_0^+}\bar\Pi_{\Theta_t}(-\xi_t)\,\mathrm{d}t\right].$$

On the other hand, by the Lévy system representation given in (2.1), we have

$$\begin{aligned}
\mathbf{P}_{y,\theta}\left(\xi_{\tau_0^+}>0\right) &= \mathbf{P}_{y,\theta}\left(\sum_{s\le\tau_0^+}\mathbb{1}_{\{\xi_s>0\}}\right)\\
&= \mathbf{P}_{y,\theta}\left[\int_0^{\tau_0^+}\mathrm{d}s\int_{\mathcal{S}\times\mathbb{R}}\mathbb{1}_{\{\xi_s+z>0\}}\Pi(\Theta_s,\mathrm{d}v,\mathrm{d}z)\right]\\
&= \mathbf{P}_{y,\theta}\left[\int_0^{\tau_0^+}\Pi(\Theta_s,\ \mathcal{S},\ (-\xi_s,+\infty))\mathrm{d}s\right]
\end{aligned}$$

(7.4)
$$= \mathbf{P}_{y,\theta}\left[\int_0^{\tau_0^+}\bar\Pi_{\Theta_s}(-\xi_s)\mathrm{d}s\right].$$

The lemma now follows by plugging (7.4) into the right-hand side of (7.3). □

Lemma 7.4. *For every nonnegative measurable function $f:\mathbb{R}\times\mathcal{S}\to\mathbb{R}^+$,*

(7.5)
$$\int_{\mathbb{R}\times\mathcal{S}}\rho_2^{\oplus}(\mathrm{d}r,\mathrm{d}\theta)\hat{\mathbf{P}}^{\downarrow}_{r,\theta}\left[\int_0^{\bar\zeta} f(\xi^\varphi_t,\Theta^\varphi_t)\,\mathrm{d}t\right]=\int_{\mathbb{R}\times\mathcal{S}}f(r,\theta)\nu_0(\mathrm{d}r,\mathrm{d}\theta)\mathbf{P}_{r,\theta}(\xi_{\tau_0^+}=0).$$

PROOF. Without loss of generality we assume that f is a nonnegative compactly supported function for which the integral in the right-hand side of (7.5) is finite. First we undo the time change and write

$$\hat{\mathbf{P}}^{\downarrow}_{r,\theta}\left[\int_0^{\bar{\zeta}} f(\xi^{\varphi}_t,\Theta^{\varphi}_t)\mathrm{d}t\right] = \hat{\mathbf{P}}^{\downarrow}_{r,\theta}\left[\int_0^{\zeta} \mathrm{e}^{\alpha\xi_t} f(\xi_t,\Theta_t)\mathrm{d}t\right].$$

Let $F(x,\theta) := \mathrm{e}^{\alpha x}\hat{H}^{+}_{\theta}(x)f(x,\theta)$ for $(x,\theta)\in\mathbb{R}\times\mathcal{S}$. By (4.2) and Fubini's theorem we have

$$\hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left[\int_0^{\zeta} \mathrm{e}^{\alpha\xi_t} f(\xi_t,\Theta_t)\mathrm{d}t\right] = \hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left[\int_0^{\zeta} \hat{H}^{+}_{\Theta_t}(\xi_t)^{-1}F(\xi_t,\Theta_t)\mathrm{d}t\right] = \frac{\hat{\mathrm{n}}^{+}_{\theta}\left[\int_0^{\zeta}F(-\epsilon_s,\nu_s)\mathrm{d}s\right]}{\hat{\mathrm{n}}^{+}_{\theta}\left(\zeta=+\infty\right)}.$$

Hence by the definition of $\rho_2^{\oplus}$ we get

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathcal{S}} \rho_2^{\oplus}(\mathrm{d}r,\mathrm{d}\theta)\hat{\mathbf{P}}^{\downarrow}_{r,\theta}\left[\int_0^{\bar{\zeta}} f(\xi^{\varphi}_t,\Theta^{\varphi}_t)\,\mathrm{d}t\right]\\
&= \frac{c_{\pi^+}}{\mu^+}\int_{\mathcal{S}}\hat{U}^{+}_{\pi}(\mathrm{d}\theta,\mathbb{R}^+)\frac{a^+(\theta)\hat{\mathrm{n}}^{+}_{\theta}\left(\zeta=+\infty\right)}{\ell^+(\theta)+\mathrm{n}^{+}_{\theta}(\zeta)}\hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left[\int_0^{\zeta}\mathrm{e}^{\alpha\xi_t}f(\xi_t,\Theta_t)\mathrm{d}t\right]\\
&= \frac{c_{\pi^+}}{\mu^+}\int_{\mathcal{S}}\hat{U}^{+}_{\pi}(\mathrm{d}\theta,\mathbb{R}^+)\frac{a^+(\theta)\hat{\mathrm{n}}^{+}_{\theta}\left[\int_0^{\zeta}F(-\epsilon_s,\nu_s)\mathrm{d}s\right]}{\ell^+(\theta)+\mathrm{n}^{+}_{\theta}(\zeta)}. \qquad (7.6)
\end{aligned}$$

On the other hand by Proposition 2.8 and Fubini's theorem we have

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathcal{S}}\nu_0(\mathrm{d}y,\mathrm{d}\theta)f(y,\theta)\mathbf{P}_{y,\theta}\left(\xi_{\tau_0^+}=0\right)\\
&= \frac{c_{\pi^+}}{\mu^+}\int_{\mathbb{R}^-\times\mathcal{S}}\mathrm{d}y\pi(\mathrm{d}\theta)\mathrm{e}^{\alpha y}\hat{H}^{+}_{\theta}(y)f(y,\theta)\mathbf{P}_{0,\theta}\left(\xi_{\tau^+_{-y}}=-y\right)\\
&= \frac{c_{\pi^+}}{\mu^+}\int_{\mathbb{R}^+\times\mathcal{S}}\mathrm{d}z\pi(\mathrm{d}\theta)\mathrm{e}^{-\alpha z}\hat{H}^{+}_{\theta}(-z)f(-z,\theta)\int_{\mathcal{S}}a^+(v)u^{+}_{\theta}(\mathrm{d}v,z)\\
&= \frac{c_{\pi^+}}{\mu^+}\int_{\mathcal{S}}\pi(\mathrm{d}\theta)\int_{\mathcal{S}\times\mathbb{R}^+}U^{+}_{\theta}(\mathrm{d}v,\mathrm{d}z)F(-z,\theta)a^+(v).
\end{aligned}$$

From this and (7.6) we can see that to show (7.5), it suffices to show
(7.7)

$$\int_{\mathcal{S}}\hat{U}^{+}_{\pi}(\mathrm{d}\theta,\mathbb{R}^+)\frac{a^+(\theta)\hat{\mathrm{n}}^{+}_{\theta}\left[\int_0^{\zeta}F(-\epsilon_s,\nu_s)\mathrm{d}s\right]}{\ell^+(\theta)+\mathrm{n}^{+}_{\theta}(\zeta)} = \int_{\mathcal{S}}\pi(\mathrm{d}\theta)\int_{\mathcal{S}\times\mathbb{R}^+}U^{+}_{\theta}(\mathrm{d}v,\mathrm{d}z)F(-z,\theta)a^+(v).$$

By Proposition 3.3 the following equation holds for all $q>0$:

$$\mathbf{P}_{0,\pi}\left[\mathrm{e}^{q\bar{g}_{e_q}}\frac{F(-\bar{\xi}_{e_q},\Theta_0)a^+(\bar{\Theta}_{e_q})}{q\left(\ell^+(\bar{\Theta}_{e_q})+\mathrm{n}^{+}_{\bar{\Theta}_{e_q}}(\zeta)\right)}\right] = \hat{\mathbf{P}}_{0,\pi}\left[\mathrm{e}^{q\left(e_q-\bar{g}_{e_q}\right)}\frac{F(-(\bar{\xi}_{e_q}-\xi_{e_q}),\Theta_{e_q})a^+(\bar{\Theta}_{e_q})}{q\left(\ell^+(\bar{\Theta}_{e_q})+\mathrm{n}^{+}_{\bar{\Theta}_{e_q}}(\zeta)\right)}\right]. \qquad (7.8)$$

By Proposition 2.3, the expectation in the left equals

$$\begin{aligned}
&\int_{\mathcal{S}}\pi(\mathrm{d}\theta)\int_{\mathcal{S}\times\mathbb{R}^+} U_\theta^+(\mathrm{d}v,\mathrm{d}z)F(-z,\theta)a^+(v)\frac{\ell^+(v)+\mathrm{n}_v^+\left(\int_0^\zeta \mathrm{e}^{-qs}\mathrm{d}s\right)}{\ell^+(v)+\mathrm{n}_v^+(\zeta)}\\
&\qquad\to \int_{\mathcal{S}}\pi(\mathrm{d}\theta)\int_{\mathcal{S}\times\mathbb{R}^+} U_\theta^+(\mathrm{d}v,\mathrm{d}z)F(-z,\theta)a^+(v) \qquad (7.9)
\end{aligned}$$

as $q\to 0+$ by the monotone convergence theorem and condition (a8) that $\mathrm{n}_v^+(\zeta)<+\infty$ for all $v\in\mathcal{S}$. Similarly by Proposition 2.3 and the monotone convergence theorem, the expectation in the right-hand side of (7.8) equals

$$\begin{aligned}
&\int_{\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^+}\hat{V}_\pi^+(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z)\mathrm{e}^{-qr}\frac{a^+(v)\hat{\mathrm{n}}_v^+\left(\int_0^\zeta F(-\epsilon_s,\nu_s)\mathrm{d}s\right)}{\ell^+(v)+\mathrm{n}_v^+(\zeta)}\\
&\qquad\to \int_{\mathcal{S}}\hat{U}_\pi^+(\mathrm{d}v,\mathbb{R}^+)\frac{a^+(v)\hat{\mathrm{n}}_v^+\left(\int_0^\zeta F(-\epsilon_s,\nu_s)\mathrm{d}s\right)}{\ell^+(v)+\mathrm{n}_v^+(\zeta)} \qquad (7.10)
\end{aligned}$$

as $q\to 0+$. Hence (7.7) follows immediately by combining (7.8)-(7.10). □

Finally, we show that the process $((\xi^\varphi,\Theta^\varphi),\hat{\mathbf{P}}^\downarrow)$ has a finite lifetime.

Lemma 7.5. *For every $x\le 0$, $\theta\in\mathcal{S}$,*

$$\hat{\mathbf{P}}^\downarrow_{x,\theta}\left(\int_0^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t<+\infty\right)=1.$$

In particular, the lifetime $\bar{\zeta}$ of the process $((\xi^\varphi,\Theta^\varphi),\hat{\mathbf{P}}^\downarrow_{x,\theta})$ is finite almost surely and $\xi^\varphi_{\bar{\zeta}-}=-\infty$ $\hat{\mathbf{P}}^\downarrow_{x,\theta}$-a.s.

Proof. Since the lifetime of the time-changed process $(\xi^\varphi,\Theta^\varphi)$ equals $\int_0^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t$, we only need to prove the first assertion. We first consider the case where $x<0$ and $\theta\in\mathcal{S}$. Recall that $\hat{\mathbf{P}}^\downarrow_{x,\theta}$ is defined from $\hat{\mathbf{P}}_{x,\theta}$ through a martingale change of measure with $W_t:=\hat{H}^+_{\Theta_t}(\xi_t)\mathbb{1}_{\{t<\tau_0^+\}}/\hat{H}^+_\theta(x)$ being the martingale. Since $\hat{H}^+_v(y)=\hat{\mathbf{P}}_{y,v}\left(\tau_0^+=+\infty\right)\in[0,1]$, W_t is a bounded $\hat{\mathbf{P}}_{x,\theta}$-martingale and hence has an almost sure limit W_∞ such that $W_t\to W_\infty$ in $L^1(\hat{\mathbf{P}}_{x,\theta})$. This implies that $\hat{\mathbf{P}}^\downarrow_{x,\theta}(A)=\hat{\mathbf{P}}_{x,\theta}\left[W_\infty\mathbb{1}_A\right]$ for all $A\in\mathcal{F}_\infty$. Hence we get

$$\hat{\mathbf{P}}^\downarrow_{x,\theta}\left(\int_0^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t<+\infty\right)=\hat{\mathbf{P}}_{x,\theta}\left[W_\infty\mathbb{1}_{\{\int_0^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t<+\infty\}}\right]. \qquad (7.11)$$

It follows by Lemma 3.3 that

$$\hat{\mathbf{P}}_{0,\pi}\left[\sup_{s\in[0,1]}|\xi_s|\right]=\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,1]}|\xi_s-\xi_1|\right]\le 2\mathbf{P}_{0,\pi}\left[\sup_{s\in[0,1]}|\xi_s|\right]<+\infty.$$

Hence the MAP $((\xi,\Theta),\hat{\mathbf{P}})$ exhibits exactly one of the tail behaviors described in Proposition 2.11. We have proved in Proposition 4.1(i) that $\hat{\mathbf{P}}_{x,\theta}\left(\tau_0^+=+\infty\right)>0$. This together with Proposition 2.10 implies that under $\hat{\mathbf{P}}_{x,\theta}$ the ordinate ξ_t drifts to $-\infty$ at a linear rate. Hence we have

$$\hat{\mathbf{P}}_{x,\theta}\left(\int_0^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t<+\infty\right)=1.$$

By this and (7.11) we get

$$\hat{\mathbf{P}}^{\downarrow}_{x,\theta}\left(\int_0^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t<+\infty\right)=\hat{\mathbf{P}}_{x,\theta}[W_\infty]=1.$$

Now we consider the case where $x=0$. We have proved in Proposition 4.3 that under $\hat{\mathbf{P}}^{\downarrow}_{0,\theta}$, ξ_t leaves 0 instantaneously and that the process $(\xi_t,\Theta_t)_{t>0}$ has the same transition rates as $((\xi_t,\Theta_t)_{t>0},\hat{\mathbf{P}}^{\downarrow}_{y,\theta})$ where $(y,\theta)\in(-\infty,0)\times\mathcal{S}$. By the Markov property we have

$$\hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left(\int_s^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t<+\infty\right)=\hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left[\hat{\mathbf{P}}^{\downarrow}_{\xi_s,\Theta_s}\left(\int_0^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t<+\infty\right)\right]$$

for any $s>0$. Hence we get $\hat{\mathbf{P}}^{\downarrow}_{0,\theta}\left(\int_0^{+\infty}\mathrm{e}^{\alpha\xi_t}\mathrm{d}t<+\infty\right)=1$. □

By Lemma 7.2 the processes $((\xi^{\varphi,\dagger},\Theta^{\varphi,\dagger}),\mathbf{P})$ and $((\xi^{\varphi},\Theta^{\varphi}),\hat{\mathbf{P}}^{\downarrow})$ are dual with respect to ν_0. By Proposition 5.4, Lemma 7.3 and Lemma 7.4 one has

$$\int_{\mathbb{R}\times\mathcal{S}}\rho^{\oplus}(\mathrm{d}r,\mathrm{d}\theta)\hat{\mathbf{P}}^{\downarrow}_{r,\theta}\left[\int_0^{\bar{\zeta}}f(\xi^{\varphi}_t,\Theta^{\varphi}_t)\mathrm{d}t\right]=\int_{\mathbb{R}\times\mathcal{S}}f(r,\theta)\nu_0(\mathrm{d}r,\mathrm{d}\theta)\tag{7.12}$$

for every nonnegative measurable function $f:\mathbb{R}\times\mathcal{S}\to\mathbb{R}^+$. We define the time-changed reversed process $(\tilde{\xi},\tilde{\Theta})$ by setting

$$(\tilde{\xi}_t,\tilde{\Theta}_t):=\left(\xi^{\varphi}_{(\bar{\zeta}-t)-},\Theta^{\varphi}_{(\bar{\zeta}-t)-}\right)\qquad\text{for }0\le t<\bar{\zeta}.$$

In view of (7.12) and Lemma 7.5 we can apply Lemma 3.2 to deduce that $((\tilde{\xi}_t,\tilde{\Theta}_t)_{0<t<\bar{\zeta}},\hat{\mathbf{P}}^{\downarrow}_{\rho^{\oplus}})$ is a right continuous strong Markov process having the same transition rates as $((\xi^{\varphi,\dagger},\Theta^{\varphi,\dagger}),\mathbf{P})$. In conclusion we have just shown the following proposition.

Proposition 7.1. *Let ϱ be the image of the probability measure $\rho^{\oplus}$ under the map $\phi:(y,\theta)\mapsto\theta\mathrm{e}^{y}$. Let $\mathbb{P}^{\searrow}_{\varrho}$ be the law of the process $(\tilde{X}_t:=\mathrm{e}^{\tilde{\xi}_t}\tilde{\Theta}_t)_{t<\bar{\zeta}}$ under $\hat{\mathbf{P}}^{\downarrow}_{\rho^{\oplus}}$. Then the process $((\tilde{X}_t)_{t<\bar{\zeta}},\mathbb{P}^{\searrow}_{\varrho})$ is a right continuous Markov process such that $\tilde{X}_0=0$ and $\tilde{X}_t\neq0$ for all $t>0$ $\mathbb{P}^{\searrow}_{\varrho}$-a.s. Moreover, $((\tilde{X}_t)_{0<t<\bar{\zeta}},\mathbb{P}^{\searrow}_{\varrho})$ is a strong Markov process having the same transition rates as the self-similar Markov process $(X,\{\mathbb{P}_z,z\in\mathcal{H}\})$ killed when exiting the unit ball.*

By applying the scaling property of ssMp, we can describe the law of the process killed when exiting the ball of radius r, for any $r>0$. Thus we see that there exists a process $(X,\mathbb{P}_0)$ started at the origin such that for any $r>0$, $((X_t)_{t<\tau_r^{\ominus}},\mathbb{P}_0)$ is equal in law to $((r\tilde{X}_{r^{-\alpha}t})_{t<r^{\alpha}\bar{\zeta}},\mathbb{P}^{\searrow}_{\varrho})$.

8. Convergence of entrance law. In the following we give a convergence lemma, which gives sufficient conditions for the candidate law $\mathbb{P}_0$ defined in Section 7 to be the weak limit of $\lim_{\mathcal{H}\ni z\to 0}\mathbb{P}_z$. The idea of its proof is from [16, Proposition 7]. For completeness we also give details here.

LEMMA 8.1. *Suppose $\{\mu_n : n \geq 0\}$ is a sequence of probability measures on $\mathcal{H}$ which converges weakly to δ_0. Then $\mathbb{P}_0 = w\text{-}\lim_{n\to+\infty}\mathbb{P}_{\mu_n}$ in the Skorokhod space if the following two conditions are satisfied:*

(i) $\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}_{\mu_n}\left[\tau_\delta^\ominus \wedge 1\right] = 0$,

(ii) there exists a $\Delta > 0$ such that for every $\delta \in (0,\Delta)$, $(X_{\tau_\delta^\ominus}, \mathbb{P}_{\mu_n}) \to (X_{\tau_\delta^\ominus}, \mathbb{P}_0)$ in distribution as $n \to +\infty$.

PROOF. Let $\mathbb{D}_{\mathbb{R}^d}$ be the space of (possibly killed) càdlàg functions $\omega : [0,\infty) \to \mathbb{R}^d$, equipped with the Skorokhod topology. We work with the Prokhorov's metric $\mathrm{d}(\cdot,\cdot)$ which is compatible with the Skorokhod convergence: for $m \in \mathbb{N}$ and two paths x, y in $\mathbb{D}_{\mathbb{R}^d}$, define

$$\mathrm{d}_m(x,y) := \inf_{\lambda\in\Lambda_m}\{\sup_{t\in[0,m]}|\lambda(t)-t| \vee \sup_{t\in[0,m]}|x(t)-y\circ\lambda(t)|\},$$

where Λ_m denotes the set of strictly increasing continuous functions $\lambda : [0,m] \to \mathbb{R}^+$ with $\lambda(0) = 0$, and define

$$\mathrm{d}(x,y) := \sum_{m=1}^{+\infty} 2^{-m}\left(\mathrm{d}_m(x,y) + \mathrm{d}_m(y,x)\right) \wedge 1.$$

To prove $\mathbb{P}_0 = \text{w-}\lim_{n\to+\infty}\mathbb{P}_{\mu_n}$ in the Skorokhod space, it suffices to prove that for an arbitrary Lipschitz continuous function $f : \mathbb{D}_{\mathbb{R}^d} \to \mathbb{R}$ with Lipschitz constant $\kappa > 0$,

$$\lim_{n\to+\infty}\mathbb{P}_{\mu_n}\left[f(X)\right] = \mathbb{P}_0\left[f(X)\right]. \tag{8.1}$$

We note that by Proposition 7.1 $\left((X_{t+\tau_\delta^\ominus})_{t\geq 0}, \mathbb{P}_0\right)$ is a Markov process having the same transition rates as $(X, \{\mathbb{P}_z, z \in \mathcal{H}\})$. In view of the Feller property of $(X, \{\mathbb{P}_z : z \in \mathcal{H}\})$ and condition (ii), [17, Theorem 4.2.5] yields that for every $\delta \in (0,\Delta)$

$$\left((X_{t+\tau_\delta^\ominus})_{t\geq 0}, \mathbb{P}_{\mu_n}\right) \to \left((X_{t+\tau_\delta^\ominus})_{t\geq 0}, \mathbb{P}_0\right)$$

in distribution under the Skorokhod topology as $n \to +\infty$. Thus by the representation theorem, there exist an appropriate probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and couplings $Y^{(n)}$, $Y^{(0)}$ of the processes $(X, \mathbb{P}_{\mu_n})$ and $(X, \mathbb{P}_0)$, respectively, such that

$$(Y^{(n)}_{t+\varsigma_n})_{t\geq 0} \to (Y^{(0)}_{t+\varsigma_0})_{t\geq 0} \quad \text{as } n \to +\infty$$

$\mathbb{P}^*$-almost surely in the Skorokhod space, where for $k \geq 1$, $\varsigma_k := \inf\{t \geq 0 : \|Y_t^{(k)}\| > \delta\}$ and $\varsigma_0 := \inf\{t \geq 0 : \|Y_t^{(0)}\| > \delta\}$. We observe that for $n \geq 1$,

$$\mathrm{d}(Y^{(n)}, Y^{(0)}) \leq 4\delta + 2\,|\varsigma_n - \varsigma_0| \wedge 1 + \mathrm{d}(Y^{(n)}_{\cdot+\varsigma_n}, Y^{(0)}_{\cdot+\varsigma_0}).$$

Thus by the Lipschitz continuity of f,

$$\left|\mathbb{P}^*\left[f(Y^{(n)})\right]-\mathbb{P}^*\left[f(Y^{(0)})\right]\right|\le 4\kappa\delta+2\kappa\mathbb{P}^*\left[|\varsigma_n-\varsigma_0|\wedge 1\right]+\kappa\mathbb{P}^*\left[\mathrm{d}(Y^{(n)}_{\varsigma_n+\cdot},Y^{(0)}_{\varsigma_n+\cdot})\right]. \tag{8.2}$$

Obviously the third term converges to 0 as $n\to+\infty$ by the dominated convergence theorem. Note that

$$\mathbb{P}^*\left[|\varsigma_n-\varsigma_0|\wedge 1\right]\le \mathbb{P}^*\left[\varsigma_n\wedge 1\right]+\mathbb{P}^*\left[\varsigma_0\wedge 1\right].$$

Condition (i) implies that

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}^*\left[\varsigma_n\wedge 1\right]=\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}_{\mu_n}\left[\tau^{\ominus}_{\delta}\wedge 1\right]=0,$$

and the right continuity of $(Y^{(0)},\mathbb{P}^*)$ implies that $\lim_{\delta\to 0}\mathbb{P}^*\left[\varsigma_0\wedge 1\right]=0$. Hence we get by (8.2) that $\limsup_{n\to+\infty}\left|\mathbb{P}^*\left[f(Y^{(n)})\right]-\mathbb{P}^*\left[f(Y^{(0)})\right]\right|\le 4\kappa\delta$. Hence (8.1) follows immediately by letting $\delta\to 0$. □

Lemma 8.2. *For any $\delta>0$ and any bounded continuous function $f:\mathcal{H}\to\mathbb{R}$, $z\mapsto \mathbb{P}_z\left[\tau^{\ominus}_{\delta}\wedge 1\right]$ and $z\mapsto\mathbb{P}_z\left[f(X_{\tau^{\ominus}_{\delta}})\right]$ are continuous on $\mathcal{H}$.*

Proof. Fix an arbitrary $\delta>0$. Suppose $z_n,z_\infty\in\mathcal{H}$ satisfies that $\lim_{n\to+\infty}z_n=z_\infty$. Since $(X,\{\mathbb{P}_z:z\in\mathcal{H}\})$ is a Feller process, by [17, Theorem 4.2.5] $(X,\mathbb{P}_{z_n})\to(X,\mathbb{P}_{z_\infty})$ in the Skorkhod space. For $n\ge 0$, let $(Y^{(n)},\mathbb{P}^*)$ and $(Y,\mathbb{P}^*)$ be couplings of $(X,\mathbb{P}_{z_n})$ and $(X,\mathbb{P}_{z_\infty})$ respectively, such that $Y^{(n)}\to Y$ $\mathbb{P}^*$-a.s. in the Skorokhod topology. Let $S:=\inf\{t\ge 0:\|Y_t\|>\delta\}$ and $\varsigma_n:=\inf\{t\ge 0:\|Y^{(n)}_t\|>\delta\}$ for $n\ge 0$. Since X is a sphere-exterior regular process, so is Y, which implies that $\|Y_t\|\ne\delta$ for any $t<S$ $\mathbb{P}^*$-a.s. In view of this, it follows by [32, Theorem 13.6.4] that

$$(\varsigma_n,Y^{(n)}_{\varsigma_n})\to(S,Y_S)\quad\mathbb{P}^*\text{-a.s.}$$

as $n\to+\infty$. Consequently $\left((\tau^{\ominus}_{\delta},X_{\tau^{\ominus}_{\delta}}),\mathbb{P}_{z_n}\right)$ converges in distribution to $\left((\tau^{\ominus}_{\delta},X_{\tau^{\ominus}_{\delta}}),\mathbb{P}_{z_\infty}\right)$, and hence this lemma follows. □

Lemma 8.3. *For any sequence $\{z_n:n\ge 0\}\subset\mathcal{H}$ with $\lim_{n\to+\infty}z_n=0$, we have*

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}_{z_n}\left[\tau^{\ominus}_{\delta}\wedge 1\right]=0. \tag{8.3}$$

Proof. By the compactness of $\mathcal{S}$, it suffices to prove (8.3) for a sequence $\{z_n:n\ge 0\}$ with $\lim_{n\to+\infty}\|z_n\|=0$ and $\lim_{n\to+\infty}\arg(z_n)=\theta$ for some $\theta\in\mathcal{S}$. We first consider the case where $\arg(z_n)=\theta$ for n sufficiently large. By Lamperti-Kiu transform one has

$$\left(\tau^{\ominus}_{\delta},\mathbb{P}_x\right)\overset{d}{=}\left(\int_0^{\tau^+_{\log\delta}}\mathrm{e}^{\alpha\xi_u}\mathrm{d}u,\mathbf{P}_{\log\|x\|,\arg(x)}\right)\quad\forall\delta>0,x\in\mathcal{H}.$$

Taking expectations of both sides and using the translation invariance of ξ and Fubini's theorem, we have for every $x \in \mathcal{H}$ with $\|x\| < \delta$,

$$
\begin{aligned}
\mathbb{P}_x[\tau_\delta^\ominus] &= \mathbf{P}_{\log\|x\|,\arg(x)}\left[\int_0^{\tau^+_{\log\delta}} \mathrm{e}^{\alpha\xi_u}\mathrm{d}u\right] \\
&= \delta^\alpha \mathbf{P}_{\log(\|x\|/\delta),\arg(x)}\left[\int_0^{\tau^+_0} \mathrm{e}^{\alpha\xi_u}\mathrm{d}u\right] \\
&= \delta^\alpha \int_0^\infty \mathrm{d}u\, \mathbf{P}_{\log(\|x\|/\delta),\arg(x)}\left[\mathrm{e}^{-\alpha(\bar{\xi}_u-\xi_u)}\mathrm{e}^{\alpha\bar{\xi}_u}\mathbb{1}_{\{\bar{\xi}_u\le 0\}}\right] \\
&= \delta^\alpha \lim_{q\downarrow 0}\frac{1}{q}\mathbf{P}_{\log(\|x\|/\delta),\arg(x)}\left[\mathrm{e}^{-\alpha(\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q})}\mathrm{e}^{\alpha\bar{\xi}_{\mathbf{e}_q}}\mathbb{1}_{\{\bar{\xi}_{\mathbf{e}_q}\le 0\}}\right]. \qquad (8.4)
\end{aligned}
$$

Set $y = \log(\|x\|/\delta) < 0$ and $u = \arg(x)$. By Proposition 2.3 and the monotone convergence theorem we have

$$
\begin{aligned}
&\frac{1}{q}\mathbf{P}_{y,u}\left[\mathrm{e}^{-\alpha(\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q})}\mathrm{e}^{\alpha\bar{\xi}_{\mathbf{e}_q}}\mathbb{1}_{\{\bar{\xi}_{\mathbf{e}_q}\le 0\}}\right] \\
&\quad= \frac{1}{q}\mathbf{P}_{0,u}\left[\mathrm{e}^{-\alpha(\bar{\xi}_{\mathbf{e}_q}-\xi_{\mathbf{e}_q})}\mathrm{e}^{\alpha\left(\bar{\xi}_{\mathbf{e}_q}-|y|\right)}\mathbb{1}_{\{\bar{\xi}_{\mathbf{e}_q}\le |y|\}}\right] \\
&\quad= \int_{\mathbb{R}^+\times\mathcal{S}\times[0,|y|]} \mathrm{e}^{-qr}\mathrm{e}^{\alpha(z-|y|)}\left[\ell^+(v)+\mathrm{n}^+_v\left(\int_0^\zeta \mathrm{e}^{-qs-\alpha s}\mathrm{d}s\right)\right]V^+_u(\mathrm{d}r,\mathrm{d}v,\mathrm{d}z) \\
&\quad\to \int_{\mathcal{S}\times[0,|y|]} \mathrm{e}^{-\alpha(|y|-z)}\left[\ell^+(v)+\mathrm{n}^+_v\left(\int_0^\zeta \mathrm{e}^{-\alpha s}\mathrm{d}s\right)\right]U^+_u(\mathrm{d}v,\mathrm{d}z) \qquad (8.5)
\end{aligned}
$$

as $q \downarrow 0$. It follows from (8.4) and (8.5) that

$$
\mathbb{P}_{z_n}\left[\tau_\delta^\ominus\right] = \delta^\alpha \int_{\mathcal{S}\times[0,|y_n|]} \mathrm{e}^{-\alpha(|y_n|-z)}\left[\ell^+(v)+\mathrm{n}^+_v\left(\int_0^\zeta \mathrm{e}^{-\alpha s}\mathrm{d}s\right)\right]U^+_\theta(\mathrm{d}v,\mathrm{d}z) \qquad (8.6)
$$

where $y_n = \log(\|z_n\|/\delta)$. Since $|y_n| \to +\infty$ as $n \to +\infty$, by (5.10) the integral in the right-hand side converges to

$$
\frac{1}{\alpha}\int_{\mathcal{S}}\left[\ell^+(v)+\mathrm{n}^+_v\left(\int_0^\zeta \mathrm{e}^{-\alpha s}\mathrm{d}s\right)\right]\pi^+(\mathrm{d}v),
$$

which is bounded from above by c_{π^+}/α. Hence (8.3) follows by letting $\delta \to 0$ in (8.6). For a more general sequence $\{z_n : n \ge 0\}$ which satisfies the conditions stated in the beginning of this proof, we set $z_n^* := \|z_n\|\theta$. The above argument shows that $\lim_{\delta\to 0}\limsup_{n\to+\infty}\mathbb{P}_{z_n^*}\left[\tau_\delta^\ominus \wedge 1\right] = 0$. Since $\lim_{n\to+\infty}\|z_n^*-z_n\| = 0$ and by Lemma 8.2 the function $z \mapsto \mathbb{P}_z\left[\tau_\delta^\ominus\wedge 1\right]$ is uniformly continuous on any compact subset of $\mathcal{H}$, we have $\lim_{n\to+\infty}\left|\mathbb{P}_{z_n^*}\left[\tau_\delta^\ominus\wedge 1\right]-\mathbb{P}_{z_n}\left[\tau_\delta^\ominus\wedge 1\right]\right| = 0$ and hence (8.3) follows. □

LEMMA 8.4. *Suppose $\{z_n : n \ge 0\} \subset \mathcal{H}$ satisfies $\lim_{n\to+\infty} z_n = 0$. Then for any $\delta > 0$, the probability measures $\mathbb{P}_{z_n}\left(X_{\tau_\delta^\ominus}\in\cdot\right)$ converges weakly to a proper distribution $\mu_\delta(\cdot)$ on $\mathcal{H}$.*

Proof. We need to show that there exists a distribution μ_δ on $\mathcal{H}$ such that

$$\lim_{n\to+\infty}\mathbb{P}_{z_n}\left[f(X_{\tau^{\ominus}_\delta})\right]=\int_{\mathcal{H}}f\mathrm{d}\mu_\delta \tag{8.7}$$

for every bounded continuous function $f:\mathcal{H}\to\mathbb{R}$. In view of Lemma 8.2 and the argument in the end of the above proof, we only need to prove that (8.7) holds for a sequence $\{z_n:n\geq 0\}$ where $\lim_{n\to+\infty}\|z_n\|=0$ and $\arg(z_n)=\theta$ for n sufficiently large. By Lamperti-Kiu transform we have

$$\begin{aligned}\mathbb{P}_{z_n}\left[f(X_{\tau^{\ominus}_\delta})\right] &= \mathbf{P}_{\log\|z_n\|,\theta}\left[f\left(\exp\{\xi_{\tau^+_{\log\delta}}\}\Theta_{\tau^+_{\log\delta}}\right)\right]\\ &= \mathbf{P}_{0,\theta}\left[f\left(\mathrm{e}^{\log\delta}\exp\{\xi_{\tau^+_{\log\frac{\delta}{\|z_n\|}}}-\log\frac{\delta}{\|z_n\|}\}\Theta_{\tau^+_{\log\frac{\delta}{\|z_n\|}}}\right)\right].\end{aligned}$$

Since $\|z_n\|\to 0$ and $\log\delta/\|z_n\|\to+\infty$, Proposition 5.3 yields that the distribution of $(\xi_{\tau^+_{\log\delta/\|z_n\|}}-\log\delta/\|z_n\|,\Theta_{\tau^+_{\log\delta/\|z_n\|}})$ converges weakly to $\rho^{\ominus}$. Thus the integral in the right-hand side of the above equation converges to $\int_{\mathbb{R}^+\times\mathcal{S}}f\left(\mathrm{e}^{\log\delta}\mathrm{e}^z v\right)\rho^{\ominus}(\mathrm{d}z,\mathrm{d}v)$. Hence (8.7) follows by setting $\mu_\delta(\cdot)=\int_{\mathbb{R}^+\times\mathcal{S}}\mathbb{1}_{\{\mathrm{e}^{\log\delta}\mathrm{e}^z v\in\cdot\}}\rho^{\ominus}(\mathrm{d}z,\mathrm{d}v)$. □

Lemma 8.5. *For any $\delta>0$, we have $\mathbb{P}_0\left(X_{\tau^{\ominus}_\delta}\in\cdot\right)=\mu_\delta(\cdot)$.*

Proof. Suppose $f:\mathcal{H}\to\mathbb{R}$ is an arbitrary bounded continuous function and $\sigma_n:=1/n$ for $n\geq 1$. By Markov property, we have for any $0<\sigma_n<\delta$,

$$\mathbb{P}_0\left[f(X_{\tau^{\ominus}_\delta})\right]=\mathbb{P}_0\left[\mathbb{P}_{X_{\tau^{\ominus}_{\sigma_n}}}\left[f(X_{\tau^{\ominus}_\delta})\right]\right]=\mathbb{P}_0\left[g(X_{\tau^{\ominus}_{\sigma_n}})\right] \tag{8.8}$$

where $g(x):=\mathbb{P}_x\left[f(X_{\tau^{\ominus}_\delta})\right]$. Since under $\mathbb{P}_0$ the process X_t leaves 0 instantaneously and continuously, we have $X_{\tau^{\ominus}_{\sigma_n}}\to 0$ $\mathbb{P}_0$-a.s. as $n\to+\infty$. Hence by Lemma 8.4, $g(X_{\tau^{\ominus}_{\sigma_n}})=\mathbf{P}_{X_{\tau^{\ominus}_{\sigma_n}}}\left[f(X_{\tau^{\ominus}_\delta})\right]\to\mu_\delta(f)$ $\mathbb{P}_0$-a.s. By letting $n\to+\infty$ in (8.8) we get that $\mathbb{P}_0\left[f(X_{\tau^{\ominus}_\delta})\right]=\mu_\delta(f)$, which yields this lemma. □

Proof of Theorem 6.1: The statements of (C1), (C2) and (C3) are from Propositions 4.2-4.3, Proposition 5.3 and Proposition 7.1, respectively. Hence we only need to show (C4) and (C5).

(C4) We get $\mathbb{P}_0=\text{w-}\lim_{\mathcal{H}\ni z\to 0}\mathbb{P}_z$ by a combination of Lemmas 8.1-8.5. Properties (4) and (5) are direct consequences of the construction of $(X,\mathbb{P}_0)$ given in Section 7. Next we show that $(X,\{\mathbb{P}_z,z\in\mathcal{H}_0\})$ is a Feller process. We use $C_\infty(\mathcal{H}_0)$ to denote the space of continuous functions on $\mathcal{H}_0$ vanishing at infinity. Fix an arbitrary $f\in C_\infty(\mathcal{H}_0)$, and let $P_tf(z):=\mathbb{P}_z\left[f(X_t)\right]$ for $z\in\mathcal{H}_0$ and $t\geq 0$. To show the Feller property, it suffices to show that $P_tf\in C_\infty(\mathcal{H}_0)$ for all $t>0$ and $\lim_{t\to 0+}P_tf(z)=f(z)$ for all $z\in\mathcal{H}_0$ (see, for example, [13, Chapter 2 section 2.2]). The latter holds naturally since $(X,\{\mathbb{P}_z,z\in\mathcal{H}_0\})$ is a right continuous process. We only need to show $P_t f \in C_\infty(\mathcal{H}_0)$ for $t > 0$. Suppose $x_n, x \in \mathcal{H}_0$ and $x_n \to x$. It is known that w-$\lim_{n\to+\infty} \mathbb{P}_{x_n} = \mathbb{P}_x$ in the Skorokhod space. If

$$\mathbb{P}_x (X_{t-} \neq X_t) = 0 \tag{8.9}$$

for $t > 0$, then it follows by [22, Proposition VI.2.1] that $(X_t, \mathbb{P}_{x_n})$ converges in distribution to $(X_t, \mathbb{P}_x)$ and hence $\lim_{n\to+\infty} P_t f(x_n) = \lim_{n\to+\infty} \mathbb{P}_{x_n} [f(X_t)] = \mathbb{P}_x [f(X_t)] = P_t f(x)$. Since $(X, \{\mathbb{P}_z, z \in \mathcal{H}\})$ is a Feller process and hence is quasi-left continuous, (8.9) holds naturally for every $x \in \mathcal{H}$ and every $t > 0$. For $x = 0$, we have by the Markov property that

$$\mathbb{P}_0 (X_{t-} \neq X_t) = \mathbb{P}_0 \left(\mathbb{P}_{X_{t/2}} \left(X_{\frac{t}{2}-} \neq X_{\frac{t}{2}} \right) \right) = 0 \quad \forall t > 0.$$

Thus we have proved (8.9) holds for all $x \in \mathcal{H}_0$ and $t > 0$. Hence $z \mapsto P_t f(z)$ is continuous on $\mathcal{H}_0$. Next we show $P_t f$ vanishes at infinity. We use $B(0, \delta)$ to denote the δ-neighborhood of 0. Since $(X, \{\mathbb{P}_z, z \in \mathcal{H}\})$ is Feller, we have

$$\lim_{\mathcal{H}\ni x\to\infty} P_t g(x) = \lim_{\mathcal{H}\ni x\to\infty} \mathbb{P}_x [g(X_t)] = 0$$

for all $g \in C_\infty(\mathcal{H}_0)$ with $g(0) = 0$. This together with the scaling property implies that the distribution of $(X_t, \mathbb{P}_x)$ converges weakly to the Dirac measure at infinity as $x \to \infty$. It follows that

$$\lim_{\mathcal{H}\ni x\to\infty} \mathbb{P}_x (X_t \in B(0, \delta)) = 0 \quad \forall \delta > 0. \tag{8.10}$$

Note that for every $x \in \mathcal{H}$ and $\delta > 0$,

$$\begin{aligned} |P_t f(x)| &\leq |\mathbb{P}_x [f(X_t); X_t \in B(0, \delta)]| + |\mathbb{P}_x [f(X_t); X_t \notin B(0, \delta)]| \\ &\leq \|f\|_\infty \mathbb{P}_x (X_t \in B(0, \delta)) + \sup_{y \in \mathcal{H}\setminus B(0,\delta)} |f(y)|. \end{aligned}$$

In view of (8.10) and the fact that f vanishes at infinity, by letting $x \to \infty$ and then $\delta \to \infty$ in the above inequality, we get that $\lim_{\mathcal{H}\ni x\to\infty} |P_t f(x)| = 0$. Hence $P_t f \in C_\infty(\mathcal{H}_0)$. Therefore $(X, \{\mathbb{P}_z, z \in \mathcal{H}_0\})$ is Feller.

Recall that $((X_t)_{t>0}, \mathbb{P}_0)$ has the same transition rates as the ssMp $(X, \{\mathbb{P}_z, z \in \mathcal{H}\})$. Thus by Markov property, to show $(X, \mathbb{P}_0)$ is a self-similar, we only need to show that $(X_t, \mathbb{P}_0) \stackrel{d}{=} (cX_{c^{-\alpha}t}, \mathbb{P}_0)$ for every $t > 0$ and $c > 0$, and this is true since

$$(X_t, \mathbb{P}_0) = \text{w-}\lim_{\mathcal{H}\ni z\to 0} (X_t, \mathbb{P}_{cz}) \stackrel{d}{=} \text{w-}\lim_{\mathcal{H}\ni z\to 0} (cX_{c^{-\alpha}t}, \mathbb{P}_z) = (cX_{c^{-\alpha}t}, \mathbb{P}_0).$$

Finally we show the uniqueness of $\mathbb{P}_0$. Suppose there exists another probability measure $\mathbb{P}_0^*$ for which the property (3) is satisfied. Using Feller property twice we get

$$\mathbb{P}_0^* (X_t \in \cdot) = \text{w-}\lim_{\mathcal{H}\ni z\to 0} \mathbb{P}_z (X_t \in \cdot) = \mathbb{P}_0 (X_t \in \cdot) \text{ for every } t > 0.$$

Hence by Markov property $\mathbb{P}_0^*$ is equal to $\mathbb{P}_0$. Suppose now that, instead, $\mathbb{P}_0^*$ satisfies the property (5). Then for any $t > 0$ and any bounded continuous function $h : \mathcal{S} \to \mathbb{R}$,

$$\begin{aligned} \mathbb{P}_0^*\left[h(X_t)\right] &= \lim_{\epsilon\to 0+} \mathbb{P}_0^*\left[h(X_{t+\epsilon})\right] \\ &= \lim_{\epsilon\to 0+} \mathbb{P}_0^*\left[\mathbb{P}_{X_\epsilon}\left[h(X_t)\right]\right] \\ &= \mathbb{P}_0\left[h(X_t)\right]. \end{aligned}$$

We used in the first equality the fact that $(X, \mathbb{P}_0^*)$ is a right continuous process and in the second equality the Markov property. The fact that $\lim_{\epsilon\to 0+} X_\epsilon = 0$ $\mathbb{P}_0^*$-a.s. and the Feller property of $(X, \{\mathbb{P}_z, z \in \mathcal{H}_0\})$ imply that $\mathbb{P}_{X_\epsilon}(X_t \in \cdot)$ converges weakly to $\mathbb{P}_0(X_t \in \cdot)$ $\mathbb{P}_0^*$-a.s. This is used in third equality. The above equation implies that $\mathbb{P}_0^*(X_t \in \cdot) = \mathbb{P}_0(X_t \in \cdot)$ for all $t > 0$, and therefore $\mathbb{P}_0^*$ is equal to $\mathbb{P}_0$ again by the Markov property.

(C5) By the strong Markov property and the sphere-exterior regularity of $(X, \{\mathbb{P}_z, z \in \mathcal{H}\})$, we have

$$\begin{aligned} \mathbb{P}_0\left(\|X_t\| = \delta \text{ for some } t \in (0, \tau_\delta^\ominus)\right) &= \mathbb{P}_0\left(\|X_t\| = \delta \text{ for some } t \in [\tau_{\delta/2}^\ominus, \tau_\delta^\ominus), \tau_{\delta/2}^\ominus < \tau_\delta^\ominus\right) \\ &= \mathbb{P}_0\left[\mathbb{P}_{X_{\tau_{\delta/2}^\ominus}}\left(\|X_t\| = \delta \text{ for some } t < \tau_\delta^\ominus\right); \tau_{\delta/2}^\ominus < \tau_\delta^\ominus\right] \\ &= 0. \end{aligned}$$

In view of this and the fact that $\text{w-lim}_{\mathcal{H}\ni z\to 0}\mathbb{P}_z = \mathbb{P}_0$ in the Skorokhod space, it follows by the Skorokhod representation theorem and [32, Theorem 13.6.4] that $\left((X_{\tau_\delta^\ominus -}, X_{\tau_\delta^\ominus}), \mathbb{P}_z\right)$ converges in distribution to $\left((X_{\tau_\delta^\ominus -}, X_{\tau_\delta^\ominus}), \mathbb{P}_0\right)$ as $z \to 0$. We note that for any $x > 0$ and $\theta \in \mathcal{S}$

$$\begin{aligned} &\mathbb{P}_{\theta e^{-x}}\left(\arg(X_{\tau_1^\ominus -}) \in dv,\ \log\|X_{\tau_1^\ominus -}\| \in dy,\ \arg(X_{\tau_1^\ominus}) \in d\phi,\ \log\|X_{\tau_1^\ominus}\| \in dz\right) \\ &= \mathbf{P}_{-x,\theta}\left(\Theta_{\tau_0^+ -} \in dv,\ \xi_{\tau_0^+ -} \in dy,\ \Theta_{\tau_0^+} \in d\phi,\ \xi_{\tau_0^+} \in dz\right) \\ &= \mathbf{P}_{0,\theta}\left(\Theta_{\tau_x^+ -} \in dv,\ \xi_{\tau_x^+ -} - x \in dy,\ \Theta_{\tau_x^+} \in d\phi,\ \xi_{\tau_x^+} - x \in dz\right). \end{aligned}$$

By Proposition 5.3 the last distribution converges weakly to $\rho(dv, dy, d\phi, dz)$ as $x \to +\infty$. Hence by the above argument we get

$$\begin{aligned} &\text{w-}\lim_{\mathcal{H}\ni z\to 0}\mathbb{P}_z\left(\arg(X_{\tau_1^\ominus -}) \in dv,\ \log\|X_{\tau_1^\ominus -}\| \in dy,\ \arg(X_{\tau_1^\ominus}) \in d\phi,\ \log\|X_{\tau_1^\ominus}\| \in dz\right) \\ &= \mathbb{P}_0\left(\arg(X_{\tau_1^\ominus -}) \in dv,\ \log\|X_{\tau_1^\ominus -}\| \in dy,\ \arg(X_{\tau_1^\ominus}) \in d\phi,\ \log\|X_{\tau_1^\ominus}\| \in dz\right) \\ &= \rho(dv, dy, d\phi, dz). \end{aligned}$$

This concludes the proof. □

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